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DYNAMICS FOR SCHRÖDINGER EVOLUTION PROBLEMS

Doutoramento em Matemática
Especialidade em Análise Matemática

Simão Fernandes Correia

Tese orientada por:
Prof. Doutor Mário Sequeira Rodrigues Figueira

Documento especialmente elaborado para a obtenção do grau de doutor

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Resumo Alargado

O objectivo deste trabalho é estudar certos aspectos dinâmicos relacionados com a equação de Schrödinger não-linear e com algumas variantes, tais como a equação de Ginzburg-Landau e a equação de Schrödinger hiperbólica. Como é claro pelos inúmeros trabalhos desenvolvidos sobre estes tópicos desde o século passado, existem múltiplas abordagens para se estudar estas equações. Uma dessas perspectivas é a de procurar certas soluções especiais (isto é, soluções da equação que satisfaçam alguma propriedade adicional) e estudar o comportamento do sistema dinâmico nas respectivas vizinhanças. Fixada esta abordagem, várias questões se colocam: será que tais soluções existem? Serão únicas? Serão estáveis? São estas as questões que estudamos neste trabalho.

Na primeira parte, consideramos um sistema de equações de Schrödinger não-lineares acopladas, que admite uma classe de soluções estacionárias (isto é, soluções com uma evolução temporal de tipo sinusoidal). O estudo destas soluções tem sido alvo de investigação activa nos últimos quinze anos, quer pelo seu interesse matemático, quer pelo interesse físico, já que o sistema em causa modela a interacção de feixes de luz monocromáticos. Neste modelo, a interacção entre dois feixes é determinada pelo coeficiente de acoplamento: os feixes exibem um comportamento atractivo se e só se o coeficiente de acoplamento for positivo. De entre as soluções estacionárias, focamo-nos sobre aquelas minimizam a acção do sistema. Estas soluções denominam-se estados de acção mínima.

Em primeiro lugar, mostramos a equivalência entre a existência de estados de acção mínima e uma desigualdade puramente algébrica envolvendo os coeficientes de acoplamento. No caso em que as frequências destas soluções são todas iguais, determinamos explicitamente todas as soluções, independentemente dos coeficientes de acoplamento. Por conseguinte, vários resultados existentes na literatura sobre o conjunto de soluções (em combinações particulares entre número de equações e os coeficientes de acoplamento) são simples corolários do nosso resultado geral.

No caso em que as frequências das componentes do estado de acção mínima não são todas iguais, uma caracterização completa do conjunto de soluções torna-se impossível. Assim sendo, viramos a nossa atenção para a questão da existência de estados de acção mínima totalmente não-triviais, isto é, soluções em que todas as suas componentes são diferentes de zero. Para abordar este problema, adoptamos três estratégias diferentes: uma técnica perturbativa nos coeficientes do sistema; a construção de uma função real de variável real cujas propriedades estão intimamente ligadas à existência de estados totalmente não-triviais; a restrição ao caso particular em que a não-linearidade do sistema é cúbica. Como resultado, obtemos diversos teoremas sobre a existência de tais soluções, consoante as frequências das componentes e os coeficientes de acoplamento.

Encerrando a primeira parte, estudamos as propriedades dinâmicas dos estados de acção mínima. Por um lado, o facto destes estados minimizarem a acção implica compor-

tamentos qualitativos globais por parte do sistema dinâmico. Por outro lado, analisamos a estabilidade destas soluções. Os resultados obtidos no caso vectorial são análogos aos do caso escalar. Contudo, certos argumentos são simplificados para se alcançar uma maior clareza da teoria.

Na segunda parte da tese, consideramos três variantes da equação de Schrödinger não-linear: a equação hiperbólica; a equação de Ginzburg-Landau complexa; uma equação abstracta com uma não-linearidade não-local. Em relação à primeira equação, as técnicas usadas no quadro da equação de Schrödinger para o estudo de propriedades qualitativas são, em geral, inutilizáveis. Assim, apresentamos várias classes de soluções particulares, para as quais a análise dos comportamentos qualitativos é possível. Entre essas soluções, destacamos as soluções de onda plana, ou seja, soluções que, nas variáveis espaciais, satisfazem uma equação do transporte linear. Em particular, construímos as ferramentas necessárias para o estudo da interacção entre soluções de onda plana e soluções localizadas e estudamos a estabilidade das primeiras quando perturbadas pelas segundas.

A equação de Ginzburg-Landau complexa pode ser vista como um caso intermédio entre a equação de Schrödinger e a equação do calor. O facto de a natureza destas duas equações ser essencialmente diferente (a primeira é uma equação Hamiltoniana, com múltiplas leis de conservação; a segunda é uma equação de difusão, com princípios do máximo) faz com que a teoria qualitativa para a equação de Ginzburg-Landau seja extremamente complexa. Neste trabalho, consideramos três problemas: a existência de soluções estacionárias em combinações específicas dos coeficientes da equação; a existência e estabilidade de soluções de onda plana em dimensão três, mostrando em particular que a perturbação localizada de soluções de onda plana origina soluções globalmente definidas; a análise qualitativa no caso particular em que não existe difusão, onde determinamos concretamente os expoentes da não-linearidade para os quais existem soluções globalmente limitadas no tempo.

Terminamos a segunda parte com o estudo de uma equação de evolução abstracta com uma não-linearidade não-local. Esta equação pode ser vista como uma simplificação de equações mais complexas, tais como a de Ginzburg-Landau. Na verdade, dada uma condição inicial, é possível obter explicitamente a solução desta equação. Este facto permite analisar de forma bastante fina o conjunto de soluções globalmente definidas, incluindo propriedades de convexidade e conexidade. Além disso, ao definirmos uma norma intrinsecamente relacionada com tal conjunto, obtemos, por fecho topológico, novos espaços funcionais, que relacionamos posteriormente com os espaços associados ao operador abstracto. Por fim, mostramos que a explicitude da solução da equação pode ser usada de forma elementar para obter resultados de existência global para equações do calor não-lineares.

Na terceira e última parte deste trabalho, obtemos novos resultados qualitativos para a equação de Schrödinger não-linear. Numa primeira fase, extendemos a teoria de soluções de onda plana para a superposição numerável de ondas planas com velocidades distintas (ou seja, satisfazendo, nas variáveis espaciais, equações do transporte distintas). Neste quadro, além da interacção entre onda plana e solução localizada, temos também a interacção entre as ondas com velocidades diferentes. De forma decisiva, vemos que esta interacção se comporta como um termo localizado, sendo incluída de forma natural na evolução da solução localizada. Assim, obtemos a existência local de solução quando o dado inicial é a soma de um dado localizado e de uma superposição numerável de ondas planas. Mais, provamos a estabilidade da superposição de ondas planas com respeito a perturbações localizadas.

Passando ao nível seguinte em complexidade, substituímos a soma por uma integração sobre todas as velocidades possíveis. Ao fazer este processo, aparece de forma natural a transformada da onda plana, cuja construção é análoga à da transformada de Fourier. Para construir a teoria dinâmica, estudamos primeiro as propriedades desta nova transformada. Em particular, vemos que o espaço das imagens desta transformada não está contido no espaço das funções de quadrado integrável, sendo por isso algo diferente do que é habitual na teoria existente para a equação de Schrödinger não-linear. Além disso, o grupo linear associado à equação comporta-se da forma esperada no espaço das imagens da transformada, permitindo assim desenvolver a teoria de existência e estabilidade de soluções neste novo espaço funcional.

Finalmente, consideramos dois novos resultados qualitativos sobre a equação de Schrödinger não-linear: a concatenação de soluções globais com decaimento e a existência de soluções ilimitadas com variância infinita. O primeiro resultado mostra que, fixados dois dados iniciais que originam soluções globais com decaimento, a concatenação de um dos dados com uma translacção do outro (por forma a estarem suficientemente "afastados") dá origem a uma nova solução global com decaimento. Em particular, por múltiplas concatenações, obtemos soluções globais da equação com normas iniciais arbitrariamente grandes. O segundo resultado é uma tentativa de perceber a necessidade da hipótese de variância finita nos resultados existentes de explosão em tempo finito. Damos uma condição suficiente para a explosão em tempo finito ou infinito, que não requer variância finita, e construímos dados iniciais satisfazendo essa condição.

Este trabalho encontra-se redigido em inglês, para que possa ser lido e usado pela comunidade científica internacional.

Palavras-chave: equação de Schrödinger não-linear, soluções especiais, estabilidade.

Abstract

In this thesis, we study some dynamical aspects related to the nonlinear Schrödinger equation (NLS) and some of its variants, such as the complex Ginzburg-Landau equation and the hyperbolic nonlinear Schrödinger equation. Our focus is the existence and stability of "special" solutions (which includes bound-states, spatial plane waves and so on). Part I is dedicated to the theory of ground-states for a system of weakly coupled nonlinear Schrödinger equations; Part II looks at three variants of the (NLS); Part III presents some new results for the (NLS).

Keywords: Schrödinger equation, special solutions, stability.

2010 AMS classification: 35B35, 35Q55, 35A01.

Contents

Part II Dynamics for some variants of the nonlinear Schrödinger equation

Part III New results for the nonlinear Schrödinger equation

Preface

The goal of this thesis is to understand dynamical behaviours of the nonlinear Schrödinger equation

$$iu_t + \Delta u + \lambda |u|^\sigma u = 0 \tag{NLS}$$

and of some of its variants. The literature available is too vast to be cited: we simply refer the monographs [12] and [70] for a glimpse on this subject. Evidently, one may adopt many different perspectives on the dynamics of these evolution problems. One of those perspectives is to understand what happens around some special solutions. By "special" we mean solutions which satisfy some extra assumption (usually, that the solutions are of a specific shape). This yields a series of questions: do these solutions exist? Are they unique? Are they stable under the dynamical flow? This point of view is the main philosophy we follow throughout this work and it will be a common theme to its three parts.

This thesis is a collection of works I have developed since late 2013, when I first went to the Université Paris 6 to work under the supervision of Thierry Cazenave. The subject of my thesis at that time was to understand the dynamics behind the complex Ginzburg-Landau equation

$$u_t = e^{i\theta}\Delta u + e^{i\gamma}|u|^\alpha u, \tag{CGL}$$

which can be seen as an equation "between" the nonlinear heat equation and the nonlinear Schrödinger equation (which had been the subject of my master's thesis). The list of goals included blowup criteria and existence of stationary states, problems which, to the present day, are still mostly unanswered. As a first step, he advised me to look at a nonlocal version of the (CGL):

$$u_t = e^{i\theta}\Delta u + e^{i\gamma}\|u\|^\alpha u,$$

where $\|u\|$ is the norm of u in some suitable functional space. It turned out that one could actually solve *explicitly* the Cauchy problem. This lead to a very precise understanding of the dynamics behind the equation, but also indicated that this nonlocal version could not mimic suitably the complex dynamics behind (CGL).

Some months later, he introduced me to two of his co-workers, Flávio Dickstein and Fred B. Weissler. Almost instantly, I was invited to participate in their joint research sessions, which was, for me, the first time cooperating in research. They were working in the limit case of (CGL) where $\theta = \pi/2$. As a result, we proved finite or infinite blow-up for any nonzero solution in $\mathbb{R}^d$.

During the time I was in Paris, I travelled back and forth to Lisbon, where my personal connections remained, and I kept in touch with my master's thesis advisor, Mário Figueira. In September 2014, I decided to return to Lisbon definitely and conclude my PhD under his supervision. This choice was made for personal reasons and also because I was feeling unacomplished with the course of my thesis. Looking back, I wonder if it wouldn't have been wiser to opt for a co-supervision (which was offered to me by Cazenave). However, this brings me no sense of regret: I am pleased with the work I have developed here in Lisbon and I was able to fully enjoy these three years.

The first subject I worked on after my return was on the existence, uniqueness and stability of minimal action solutions of the stationary system

$$\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} k_{jm} |u_j|^{p+1} |u_m|^{p-1} u_m = 0, \quad m = 1, ..., M, \tag{M-NLS}$$

which is equivalent to the study of ground-states of a certain system of M weakly coupled nonlinear Schrödinger equations. I quickly discovered that such a problem had been, for the last decade, an intensive field of research (motivated mostly by the system itself and not by the fact that it yields ground-states of a dynamical problem). Regarding this subject, I soon obtained several results, especially on the coherent case $\omega_m \equiv \omega$.

A few months after, during a routine catchup of the latest publications on arXiv, I noticed that Filipe Oliveira, a professor at Universidade Nova de Lisboa who had been a member of my master's thesis jury, had written a paper on the same subject. I immediately contacted him and we started working together. He then told me he was also working with Hugo Tavares, a young researcher at Instituto Superior Técnico in Lisbon (who, coincidentally, was also a jury at my master's thesis). The three of us got together and wrote a nice paper regarding the incoherent case when $p = 1$.

After this incursion into the elliptic world, my supervisor suggested we could work on the hyperbolic nonlinear Schrödinger equation

$$iu_t + u_{xx} - u_{yy} + \lambda |u|^\sigma u = 0. \tag{HNLS}$$

The nonellipticity of differential operator makes many of the fine techniques available for the (NLS) unusable. In the lines of our general philosophy and also motivated by physical interpretations, we looked for some particular solutions and studied their stability properties. Among them, we highlight the *spatial plane wave* solutions, which are of the form $u(t, x, y) = f(t, x - cy)$, where $c \in \mathbb{R}$ is called the *speed* of the wave. We developed a suitable framework to study the effect of localized perturbations on these non-integrable solutions and proved some stability results.

Motivated by the simple form of the spatial plane wave solutions and by the results we had obtained, we further developed the theory, this time in the context of the (NLS). First, we studied what happens when one superposes a numerable sequence of plane waves with different speeds. Even though the technicalities were a bit more involved, the results we obtained were the expected generalizations of the single wave case. It then occured to me that one might be able to replace the numerable sum by a continuous integration. As soon as we started to work with this continuous case, we understood that it was intrinsically different from the numerable one. Fortunately, it turned out to produce a very interesting theory, which is based on the *plane wave transform*

$$(Tf)(x, y) = \int f(x - cy, c)dc.$$

Trying to test the limits of this technique, my advisor suggested we could study the same problem for the incompressible Navier-Stokes equation in dimension three. Even though my knowledge regarding this equation is fairly minimal, we were able to show the stability of spatial plane waves (which, in the fluids community, correspond to *laminar* fluids). Afterwards, we noticed that the (CGL) shared several properties with incompressible Navier-Stokes and so an analogous result was shown for the (CGL).

Finally, during some investigative free time, I tried to obtain some qualitative results for the (NLS): on one hand, to evaluate the necessity of the finite-variance requirement for the blow-up of negative energy solutions; on the other, to understand the implications of an inequality which I now call *finite speed of disturbance* (as a weaker notion of finite speed of propagation).

We now briefly explain the structure of this thesis. The first part concerns the theory for the (M-NLS) system: after a more detailed introduction to subject (Chapter 1), we show existence of ground-states and fully characterize the set of ground-states in the coherent case (Chapter 2). The incoherent case, where such a characterization is unavailable, is then the subject of Chapter 3. Finally, in Chapter 4, we present some dynamical results which are connected with the ground-states: on one hand, we have general qualitative results related to the minimal action property of the ground-states; on the other, we study the stability of bound-states through the dynamical flow.

In the second part, we focus on three different variants of the (NLS), namely the hyperbolic nonlinear Schrödinger equation (Chapter 5), the complex Ginzburg-Landau equation (Chapter 6) and the nonlocal version of the (CGL) (Chapter 7). The first two chapters will deal mostly with the existence of special solutions and their stability properties. In Chapter 7, a more complete study of the evolution problem is carried out.

Finally, in the third part, we present the new results for the nonlinear Schrödinger equation: the general theory of spatial plane waves in both numerable and continuous cases (Chapter 8); two results regarding boundedness or unboundedness of solutions to the (NLS) (Chapter 9).

It is worth noticing that, while Part I should be read as is, the order of the chapters of Parts II and III is not very important, since they are mostly self-contained. Moreover, this work is essentially a compilation (with few minor adjustments and improvements) of the papers [13], [19], [20], [21], [22], [23], [24], [25] and [26].

I end this preface with the deserved acknowledgements. This work was developed with the support of the Fundação para a Ciência e Tecnologia, through the PhD grant

SFRH/BD/96399/2013 and also through the grant UID/MAT/04561/2013. I am most indebted to Mário Figueira, for his guidance and solicitude throughout these years, and to Thierry Cazenave, who accepted to be my PhD supervisor and has always been available for discussions even after my return to Lisbon. I want to thank to the several professors with whom I had the pleasure to work with in the last four years: Flávio Dickstein, Fred B. Weissler, Filipe Oliveira, Hugo Tavares, Rémi Carles, Jorge Drummond da Silva and Ádan Corcho (in the context of a CAPES-FCT research project). In a personal point of view, I thank my PhD colleagues, with whom I shared many of the good (and also the hard) times of the research process. I thank my family for their warm support and for always believing in my success (in any greater lenght than the one I had for myself). Last, but not least, I thank my wife Andreia, who has always been by my side through better or worse, and whose love nurtured me to become who I am today. Without it, I am certain I would not have been able to pursue the research as I have in these past years.

Lisbon, May 28th 2017

Notation

a.e.	almost everywhere
$\operatorname{Re} z$	the real part of a complex number z
$\operatorname{Im} z$	the imaginary part of a complex number z
$\bar{z}$	the complex conjugate of a complex number z
$\overline{Y}$	the closure in X of a topological subspace Y
X'	the topological dual of a space X
X^M	the product of M copies of a topological space X
$\langle u, v\rangle_{X'\times X}$	the duality product of $u \in X'$ with $v \in X$
$X \hookrightarrow Y$	$X$is a subset of Y with continuous injection
$u_n \rightharpoonup u$ in E	u_n converges to u in the weak topology of the Banach space E
$\operatorname{dist}(A, B)$	the Euclidean distance between the sets $A, B \subset \mathbb{R}^d$
u_x	the partial derivative $\frac{\partial u}{\partial x}$
∇u	the vector of all partial derivatives of u
Δu	the trace of the Hessian matrix of u
$\mathcal{D}(\Omega)$	the space of infinitely differentiable functions with compact support in Ω
$\mathcal{S}(\mathbb{R}^d)$	the Schwarz space of rapidly decaying functions in $\mathbb{R}^d$
$\mathcal{F}u$	the Fourier transform of a function u
$L^p(\Omega, X)$	the space of measurable functions u in Ω with values in X such that $\\|u\\|_{L^p(\Omega,X)} = \left(\int_\Omega \\|u(x)\\|_X^p dx\right)^{\frac{1}{p}} < \infty, \text{ if } p < \infty,$ $\\|u\\|_{L^p(\Omega,X)} = \text{ ess } \sup_{x\in\Omega} \\|u(x)\\|_X < \infty, \text{ if } p = \infty$
$L^p(\Omega)$	$= L^p(\Omega, \mathbb{C})$
$\\|u\\|_p$	the L^p norm of a function $u \in L^p(\Omega)$
p'	the conjugate exponent of $p \in [1, \infty]$ so that $\frac{1}{p} + \frac{1}{p'} = 1$
$W^{k,p}(\Omega)$	the Sobolev space of $L^p(\Omega)$ functions whose derivatives up to order k are also in $L^p(\Omega)$
$H^k(\Omega)$	$= W^{k,2}(\Omega)$
$C^k(I, X)$	the space of functions over an interval I with values in a Banach space X which are k times differentiable
x^+	the minimum between 0 and a real number x
$1/x^+$	$1/x$ if $x > 0$, ∞ otherwise
$x \lesssim y$	there exists an absolute constant $C > 0$ such that $x \leqslant Cy$
$x \ll y$	there exists a small absolute constant $\epsilon > 0$ such that $x \leqslant \epsilon y$
$\int u dx$	the integral of u on the x variable over the whole domain of definition

Part I

Bound-states for systems of weakly coupled nonlinear Schrödinger equations

Chapter 1

Introduction and preliminaries

1.1 Introduction

Throughout Part I of this work, we shall focus on the system of M coupled semilinear Schrödinger equations

$$i(v_m)_t + \Delta v_m + \sum_{j=1}^{M} k_{jm}|v_j|^{p+1}|v_m|^{p-1}v_m = 0, \quad m = 1, ..., M \qquad \text{(M-NLS}_t\text{)}$$

where $\mathbf{v} = (v_1, ..., v_M) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}^M$, $k_{jm} \in \mathbb{R}$, $k_{jm} = k_{mj}$, and $0 \leqslant p < 2/(d-2)^+$.

This system arises in several physical contexts, such as nonlinear optics and Bose-Einstein condensates ([72]). From the point of view of optics, one takes $d = 2$ and makes the following interpretation: each v_m corresponds to the amplitude of a monochromatic[1] electric field (or simply *beam*) propagating in the t-direction[2]; $x \in \mathbb{R}^2$ is the coordinate of the plane perpendicular to the axis of propagation; each coupling coefficient k_{jm} is connected to the refraction index of beam m passing through beam j. Positive couplings mean that the beam j forces beam m to concentrate more along the axis of propagation, while negative couplings imply a repeling behaviour. As such, we say that positive couplings are attractive and negative ones are repulsive (see [66] and the physical references therein).

One of the problems one may study is the local Cauchy problem for this system, which prescribes the amplitudes of each beam at $t = 0$. Following the same arguments as for the (NLS), one may prove the standard H^1 local well-posedness result:

Proposition 1.1.1. *Fix $p \geqslant 1$. Given $\mathbf{v_0} \in (H^1(\mathbb{R}^d))^M$, there exist $T(\mathbf{v_0}) > 0$ and a unique maximal solution $\mathbf{v} \in C([0, T(\mathbf{v_0})); (H^1(\mathbb{R}^d))^M)$ of* (M-NLS$_t$)*, which depends continuously on the initial data. If $T(\mathbf{v_0}) < \infty$, then one has the blow-up alternative*

$$\lim_{t \to T(\mathbf{v}_0)} \|\nabla \mathbf{v}(t)\|_2 = +\infty.$$

[1] This means that the electric field has the form E(spatial variables)exp($i\times$frequency$\times$time), thus one must simply determine E.

[2] Hence t is viewed as a spatial variable.

REMARK 1.1.1. The condition $p \geqslant 1$ guarantees that the nonlinearity is locally Lipschitz, which is an important feature in order to apply the usual techniques of local well-posedness. The local existence for $0 < p < 1$ is an open problem, possibly solvable using some topological techniques (instead of fixed-point arguments). However, the remainder of the theory we shall develop throughout these chapters works for $p > 0$ without any further restrictions.

Having the question of local existence settled, our goal will be to understand some qualitative behaviours for this Cauchy problem. Naturally, many of the properties of the (NLS) can be translated to system (M-NLS$_t$). For example, one has the conservation of individual mass (or charge)

$$M_m(\mathbf{v(t)}) = M_m(\mathbf{v_0}) := \|v_m\|_2^2, \quad m = 1, \dots, M,$$

and the conservation of total energy

$$E(\mathbf{v(t)}) = E(\mathbf{v_0}) := \frac{1}{2}\sum_{m=1}^{M} \|\nabla v_m\|_2^2 - \frac{1}{2p+2}\sum_{j,m=1}^{M} k_{jm}\|v_j v_m\|_{p+1}^{p+1}.$$

As a consequence, there are several qualitative results that are valid for system (M-NLS$_t$), simply by following the arguments for the (NLS). Global existence for $p < 2/d$ or Virial's argument for blowup of solutions with negative energy are some easy examples, which we shall not prove. This kind of results does not depend heavily on the number of equations and so we do not observe new dynamical behaviour in this way.

Following the general principle discussed in the preface and the known theory for the (NLS), we look for nontrivial periodic solutions of the form $\mathbf{v} = e^{i\omega t}\mathbf{u}$, with $\omega > 0$ and $\mathbf{u} = (u_1, ..., u_M) \in (H^1(\mathbb{R}^d))^M \backslash \{0\}$. We call these solutions *bound-states*[3]. This ansatz leads us to study the system

$$\Delta u_m - \omega u_m + \sum_{j=1}^{M} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0, \quad m = 1, ..., M. \tag{M-NLS}$$

These solutions have the particular property of perfectly balancing the diffraction of the beams (associated to the Δu_m term) and the refraction effects. A solution of (M-NLS) can be seen to be a critical point of the action functional

$$S(\mathbf{u}) := \frac{\omega}{2}\sum_{m=1}^{M} \|u_m\|_2^2 + E(\mathbf{u}), \quad \mathbf{u} \in (H^1(\mathbb{R}^d))^M.$$

Especially relevant, for both physical and mathematical reasons, are the bound-states which have minimal action among all bound-states, the so-called *ground-states*. We set

$$G = \{\mathbf{u} \text{ bound-state} : S(\mathbf{u}) \leqslant S(\mathbf{w}), \text{ for all } \mathbf{w} \text{ bound-state}\}.$$

[3]This denomination will be used both for the periodic solution $\mathbf{v}$ and for the spatial profile $\mathbf{u}$.

In the scalar case, one may prove that there is a unique ground-state (modulo translations and rotations). The properties of the ground-state determine qualitative behaviours for the dynamical system (NLS). For example, in the L^2-critical case ($\sigma = 4/d$), the ground-state determines the minimal mass for the existence of blow-up, quantization of mass at blow-up, minimal blow-up solutions, etc. Examples of its relevance may be found, for example, in [75], [57], [12].

The vector-valued case has attracted a lot of attention in the past ten years. More precisely, most of the works are related to the existence of *non-degenerate* ground-states, which minimize the action only among bound-states that have all components different from zero (see, for example, [2], [4], [47]). In fact, there are almost no results concerning ground-states and their characterization, which is the main problem on which we shall focus throughout Chapter 2. To our knowledge, only the papers [54], [74], [47] present advances in the characterization of ground-states, where the results obtained are quite specific. The approach for the first two is an analysis of the system of ODE's that one obtains after proving that all ground-states are radial functions. In the third paper, the approach is variational and offers only conditions for the existence (or nonexistence) of ground-states with all components different from zero. However, each of these results displays several restrictions, both on the power p and on the coefficients k_{jm}.

The first question one should ask is: *are there any ground-states*? This question will be answered completely in Section 2.1. It is quite straightforward to obtain a necessary condition for the existence of bound-states: multiplying each equation by $\bar{u}_m$, integrating over $\mathbb{R}^d$ and summing in m,

$$\sum_{m=1}^{M} \omega\|u_m\|_2^2 + \|\nabla u_m\|_2^2 = \sum_{j,m=1}^{M} k_{jm}\|u_m u_j\|_{p+1}^{p+1}.$$

Hence, if a bound-state exists,

$$\left\{ \mathbf{u} \in (H^1(\mathbb{R}^d))^M : \sum_{j,m=1}^{M} k_{jm}\|u_m u_j\|_{p+1}^{p+1} > 0 \right\} \neq \emptyset.$$

This condition depends only on the coupling coefficients and expresses the fact that, if there is no possibility for the attraction effects to become larger than the repulsive ones, then the nonlinear part (which corresponds to refraction) cannot oppose the diffraction. This makes the balance needed for the existence of bound-states impossible. Even though this is *a priori* only a necessary condition, it turns out that it is equivalent to the existence of ground-states[4]. This is achieved by proving that ground-states are the solutions of the minimization problem

$$\min \left\{ \sum_{m=1}^{M} \omega\|u_m\|_2^2 + \|\nabla u_m\|_2^2 : \sum_{j,m=1}^{M} k_{jm}\|u_m u_j\|_{p+1}^{p+1} = c \right\} \tag{1.1.1}$$

[4]Hence the existence of ground-states is also equivalent to the existence of bound-states.

for some appropriate $c > 0$ and then showing the existence of minimizers. The main difference regarding the known existence results is that, instead of using Schwarz symmetrization (for which one needs the positivity of the coupling coefficients), one uses the concentration-compactness principle by P.-L. Lions (see [49], [50]). Notice that this approach does not say whether there exist radial ground-states or not. However, in conjuction with other results, we prove radiality of ground-states in all the cases where it would be possible to use Schwarz symmetrization.

On a second step, one may pose more complex questions: is the ground-state unique? Is any ground-state *fully nontrivial*, that is, are all its components nonzero? Can one characterize (explicitly) the family of ground-states? These questions are the core of Chapters 2 and 3. Throughout Part I, we write the set of fully nontrivial ground-states as

$$G^* = \{\mathbf{u} \in G : u_m \not\equiv 0,\ m = 1, \dots, M\}.$$

If a ground-state is not fully nontrivial, then it is *semitrivial*. Evidently, if one finds a complete explicit characterization of G, then the nontriviality of ground-states can be easily studied. In Section 2.2, we study the characterization problem. This was tackled in [54] and [74] in very specific cases. On a first attempt to solve this issue, we managed to characterize all ground-states if the system (M-NLS) has the following property: it is possible to group the components in such a way that two components attract each other if and only if they are in the same group. This property is verified in all the previously referred papers. Intuitively, the results tell us that the attractive components have the same profile and, if there are two repulsive components, one of them has to be zero: otherwise, it would be possible to move them away from each other indefinitely and therefore lower the action, which would contradict the minimality of the ground-state. If this grouping hypothesis fails, the situation becomes much more difficult. The reason is that two components may repel each other directly but, by transitivity, they may also attract each other (see Figure 1.1). Then the balance between these forces is not clear and the analysis is not straightforward.

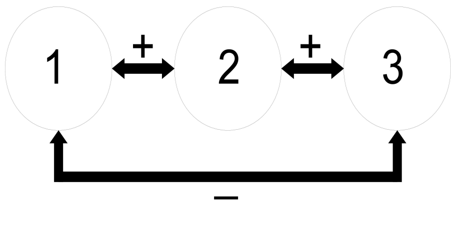

Figure 1.1: The simplest balanced system: the signs indicate whether the components attract or repel each other. Though components 1 and 3 repel each other, they are both attracted to component 2. This case was studied for the first time in [47].

One of the main properties of the minimization problem (1.1.1) is that it involves two homogeneous functionals. As a consequence, problem (1.1.1) is equivalent to

$$\max\left\{\sum_{j,m=1}^{M} k_{jm}\|u_m u_j\|_{p+1}^{p+1} : \sum_{m=1}^{M}\|u_m\|_2^2+\|\nabla u_m\|_2^2=c'\right\}.$$

Hence we must maximize the nonlinear part over a ball in $(H^1(\mathbb{R}^d))^M$. The key insight is that the homogeneity of the functionals involved seems to indicate that it is not important the value of each u_m at each specific point $x \in \mathbb{R}^d$, but the overall size of u_m (this intuition points in the right direction, even though more conditions on the functionals involved are needed). What this means is that the maximization problem should somehow be related to the finite-dimensional problem

$$\max\left\{\sum_{j,m=1}^{M} k_{jm}x_m^{p+1}x_j^{p+1} : x_m \in \mathbb{R}_0^+, \sum_{m=1}^{M} x_m^2 = 1\right\},$$

which determines the proportionality constants between different components[5]. This is indeed true (see Theorem 2.2.4) for any choice of couplings. Thus the characterization question is completely answered (and the nontriviality of ground-states follows).

From these general characterization theorems, one may obtain as simple corollaries the results of [54], [74] and [47]. Moreover, we compute explicitly the set of ground-states for some particular systems to get an insight on the complex relations between coupling coefficients and the existence of fully nontrivial ground-states. This is the subject of Section 2.3.

On the other hand, one may also consider periodic solutions of (M-NLS$_t$) where the time-frequency is not necessarily the same for each component (one then says that the components are *incoherent*). These solutions are of the form

$$\mathbf{v} = (e^{i\omega_1 t}u_1, ..., e^{i\omega_M t}u_M), \quad u_m \in H^1(\mathbb{R}^d),\ \omega_m > 0,$$

and the corresponding stationary system is

$$\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0, m = 1, ..., M. \qquad \text{(M-NLS')}$$

As such, one may study the existence and characterization of incoherent ground-states, following the lines of Chapter 2. On one hand, the existence result is extendible to this situation with almost no changes. On the other hand, the characterization results fail completely: it is actually impossible to have proportional components when the frequencies are not all the same. Therefore, in this framework, we shall move our attention to the existence of fully nontrivial ground-states, which will be the focus of Chapter 3. In this framework, we advise the reader to check [55], [51], [56] (and references therein). To tackle this problem, we adopt three different approaches: a perturbation theory, an extension of Mandel's characteristic function (see [56]) and the case $p = 1$.

[5] Therefore all components should have the same profile.

First approach. We start by explaining the perturbative method. First of all, a scaling reduces any (M-NLS') system to the case $\omega_m \geqslant 1$. Given a nonempty symmetric subset P of $\{1, ..., M\}^2$, $\beta \in \mathbb{R}$ and $\eta > 0$, consider, for $m = 1, ..., M$,

$$\Delta u_m - (1 + \eta(\omega_m - 1))u_m + \sum_{(j,m) \notin P} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m + \sum_{(j,m) \in P} \beta k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0.$$

For the sake of simplicity, suppose that $k_{jm} > 0$, for all (j, m) in P. If one considers the ground-state action level, $\mathcal{I}_\beta^\eta$, and the semitrivial ground-state action level, $(\mathcal{I}_\beta^\eta)^{sem}$, then $\mathcal{I}_\beta^\eta < (\mathcal{I}_\beta^\eta)^{sem}$ is equivalent to $G = G^*$. The continuity of these action levels with respect to β and η leads to perturbation results: if, for some β_0, η_0, one proves that the ground-state action level is strictly lower than the semitrivial action level, then the same inequality is valid for β, η close to β_0, η_0.

Second approach. For the approach using the ideas of R. Mandel, given a nonempty symmetric subset P of $\{1, ..., M\}^2$ and $\beta \in \mathbb{R}$, consider the system

$$\Delta u_m - \omega_m u_m + \sum_{(j,m) \notin P} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m + \sum_{(j,m) \in P} \beta k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0.$$

We shall build a mapping $\beta \mapsto \hat{\beta}$ such that, if

- $\beta < \hat{\beta}$, then no ground-state is fully nontrivial ($G^* = \varnothing$);
- $\beta > \hat{\beta}$, then all ground-states are fully nontrivial ($G^* = G$);
- $\hat{\beta} = \beta$, then some ground-states are semitrivial ($G \backslash G^* \neq \varnothing$).

This mapping was introduced in [56] for the system with two equations to study the existence of nontrivial ground-states as a function of the coupling coefficient k_{12}. The (very important) feature of the case $M = 2$ is that semitrivial bound-states are never influenced by the coupling coefficient. This implies that $\hat{\beta}$ is constant and therefore it defines in a very precise way when G^* is non-empty. For more than two equations, $\hat{\beta}$ is not that well-behaved, as one may observe in Section 2.3.

Third approach. Finally, we consider the cubic nonlinearity case, that is, $p = 1$ (for previous works in this problematic in the case $M = 2$, we refer to [1], [2], [47], [66], [56] and [55]. The case $M \geqslant 3$ was also studied in very particular cases in [52], [51], [16], [62] and [67]). In Section 3.3, we state qualitatively what kind of combinations on the parameters give rise either to semitrivial or to fully nontrivial ground states. In particular, it will be evident from our analysis that the different families of parameters play distinct roles: while the choice of the k_{mm} coefficients can be somewhat arbitrary, only some combinations between different ω_m, and also between different k_{jm}, $j \neq m$, allow fully nontrivial ground states to arise.

We split our results in two groups: existence results (Subsection 3.3.1) on one hand and nonexistence results (Subsection 3.3.2) on the other. We briefly summarize the main ideas:

- In terms of the ω_m coefficients, in order to get fully nontrivial ground states, the two lowest coefficients (say ω_1 and ω_2, $\omega_1 \leqslant \omega_2$) can be chosen arbitrarily, while the remaining (ω_m, $m \geqslant 3$) cannot lie too far apart from ω_2.
  Heuristically, larger ω_m make the action larger. Therefore the conclusion is that the components u_1 and u_2 of the ground states (which are the components associated to ω_1 and ω_2) can always "survive" (i.e. $u_1, u_2 \not\equiv 0$) when these parameters increase, while the non-nullity of the remaining components $u_m,\ m \geqslant 3$, will depend on how large the respective ω_m/ω_2 quotients are.

- In terms of the coefficients $k_{jm}, j \neq m$: on one hand, if, for some m_0, all k_{jm}, with $m, j \neq m_0$, are sufficiently large compared with $k_{jm_0},\ j = 1, \ldots, M$, then $G^* = \varnothing$. On the other hand, if all $k_{jm}, j \neq m$, are large and close to each other, then all ground-states are fully nontrivial. Notice that these results are (qualitatively) complementary for large k_{jm}. For small k_{jm}, the ground-states are always semitrivial.

We end Part I with the dynamical implications that ground-states have on the time-dependent system (M-NLS$_t$). These implications are twofold: on one hand, ground-states determine minimal blow-up norms, concentration of mass at the blow-up time and blow-up profile. On the other, one may study stability properties of these periodic solutions. As a by-product, we relate ground-states with the optimal constant of the vector-valued Gagliardo-Nirenberg inequality

$$\sum_{j,m=1}^{M} k_{jm}\|u_m u_j\|_{p+1}^{p+1} \leqslant C \left(\sum_{m=1}^{M} \omega_m \|u_m\|_2^2\right)^{p+1-\frac{dp}{2}} \left(\sum_{m=1}^{M} \|\nabla u_m\|_2^2\right)^{\frac{dp}{2}}, \ \mathbf{u} \in (H^1(\mathbb{R}^d))^M$$

and, using the optimal constant for the scalar case $M = 1$, we compute explicitly its value in the coherent case $\omega_1 = \cdots = \omega_M$.

1.2 Preliminaries

We start out with some definitions and known results which will be essential throughout Part I.

Definition 1.2.1. *(Bound-states and ground-states of* (M-NLS')*) Fix $M \geqslant 1$, $0 < p \leqslant 2/(d-2)^+$ and $\omega = (\omega_1, \ldots, \omega_M) \in \mathbb{R}^+$.*

1. *We define bound-state of* (M-NLS') *as any element $(u_1, ..., u_M) \in (H^1(\mathbb{R}^d))^M \backslash \{0\}$ solution of* (M-NLS') *and define A to be the set of all bound-states of* (M-NLS').

2. *A fully nontrivial bound-state is a bound-state such that* $u_m \not\equiv 0$, $\forall m$. *The set of such bound-states is called* A^*.

3. *Given* $\mathbf{u} = (u_1, ..., u_M) \in (H^1(\mathbb{R}^d))^M$, *set*

$$I(\mathbf{u}) = \sum_{m=1}^{M} \int |\nabla u_m|^2 + \omega_m \int |u_m|^2, \ J(\mathbf{u}) = \sum_{j,m=1}^{M} k_{jm} \int |u_m|^{p+1} |u_j|^{p+1}$$

and define the action of $\mathbf{u}$,

$$S(\mathbf{u}) = \frac{1}{2} I(\mathbf{u}) - \frac{1}{2p+2} J(\mathbf{u}).$$

4. *The set of ground-states of* (M-NLS') *is defined as*

$$G = \{\mathbf{u} \in A : S(\mathbf{u}) \leqslant S(\mathbf{w}), \ \forall \mathbf{w} \in A\} \subset A,$$

and the set of fully nontrivial ground-states is

$$G^* = G \cap A^*.$$

Remark 1.2.1. If $\mathbf{u} \in A$, $I(\mathbf{u}) = J(\mathbf{u})$ (one multiplies the i-th equation by u_m and integrates over $\mathbb{R}^d$). Therefore

$$S(\mathbf{u}) = \left(\frac{1}{2} - \frac{1}{2p+2}\right) I(\mathbf{u}) = \left(\frac{1}{2} - \frac{1}{2p+2}\right) J(\mathbf{u}).$$

Hence a ground-state is a bound-state with I (or J) minimal.

Remark 1.2.2. Throughout this work, we shall assume that k_{jm} are such that

$$\{\mathbf{u} \in (H^1(\mathbb{R}^d))^M : J(\mathbf{u}) > 0\} \neq \varnothing.$$

This hypothesis is necessary for the existence of bound-states, since $J(\mathbf{u}) = I(\mathbf{u}) > 0$, for any $\mathbf{u} \in A$.

The following two lemmata are well-known results concerning ground-states for (1-NLS) (see [12, Theorems 8.1.4, 8.1.5, 8.1.6] and [50]).

Lemma 1.2.2. *There exists* $Q \in H^1(\mathbb{R}^d)\backslash\{0\}$ *radial, positive and strictly decreasing such that*

$$G_{(1-NLS)} = \{e^{i\theta} Q(\cdot + y) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^d\}.$$

Lemma 1.2.3. $G_{(1-NLS)}$ *is the set of solutions of the minimization problem*

$$I_1(u) = \min_{J_1(w) = J_1(Q)} I_1(w), \quad J_1(u) = J_1(Q).$$

Finally, we state two results for the (M-NLS') which we do not prove. These results are natural extensions of their scalar versions and the proofs of the scalar case may be easily adapted to the vector-valued case.

Proposition 1.2.4 (Pohozaev's Identity)**.** *If* $\mathbf{u} \in A$*, then*

$$\sum_{m=1}^{M} \|\nabla u_m\|_2^2 = \frac{dp}{2p+2} J(\mathbf{u}).$$

Proposition 1.2.5 (Regularity)**.** *If* $\mathbf{u} \in A$*, then* $\mathbf{u} \in C^2(\mathbb{R}^d)$ *and there exists* $\epsilon > 0$ *such that* $e^{\epsilon|x|}|(\mathbf{u}(x)| + |\nabla \mathbf{u}(x)|) \in L^\infty(\mathbb{R}^d)$*.*

Chapter 2

Existence and characterization of ground-states of (M-NLS)

In this chapter, we study the existence and characterization problems for the system

$$\Delta u_m - \omega u_m + \sum_{j=1}^{M} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0, \quad m = 1, ..., M, \tag{M-NLS}$$

where $\omega > 0$, $0 < p < 2/(d-2)^+$ and $k_{jm} = k_{mj} \in \mathbb{R}$.

2.1 Ground-states: existence result

In this section, we prove the existence of ground-states:

Theorem 2.1.1. *The set of ground-states for* (M-NLS) *is nonempty if and only if one has the positivity condition*

$$\{\mathbf{u} \in (H^1(\mathbb{R}^d))^M : J(\mathbf{u}) > 0\} \neq \varnothing. \tag{PC}$$

On a first step, we shall prove that the set of ground-states is the set of solutions of a specific minimization problem. Afterwards, we prove that such a problem has a solution if and only if the positivity condition is valid.

Remark 2.1.1. Given $\lambda > 0$, let

$$I^\lambda = \inf_{J(\mathbf{u})=\lambda} I(\mathbf{u}) \geqslant 0.$$

By the homogeneity property of I and J, one easily checks that $I^\lambda = \lambda^{\frac{1}{p+1}} I^1$.

Remark 2.1.2. From Hölder's and Gagliardo-Nirenberg's inequality,

$$J(\mathbf{u}) \lesssim \left(\sum_{m=1}^{M} \omega\|u_m\|_2^2\right)^{p+1-\frac{dp}{2}} \left(\sum_{m=1}^{M} \|\nabla u_m\|_2^2\right)^{\frac{dp}{2}} \lesssim I(\mathbf{u})^{p+1}, \ \mathbf{u} \in (H^1(\mathbb{R}^d))^M.$$

Therefore $I^\lambda > 0$ for any $\lambda > 0$.

Since any ground-state $\mathbf{u} \in G$ satisfies $I(\mathbf{u}) = J(\mathbf{u})$, one should pick λ such that $I^\lambda = \lambda$ (so that minimizers satisfy the same condition). A simple computation determines such λ:

$$\lambda_G := \left(\inf_{J(\mathbf{u})=1} I(\mathbf{u}) \right)^{\frac{p+1}{p}}. \tag{2.1.1}$$

Lemma 2.1.2. *The minimization problems*

$$I(\mathbf{u}) = \min_{J(\mathbf{w})=\lambda_G} I(\mathbf{w}), \quad J(\mathbf{u}) = \lambda_G \tag{Min}$$

and

$$I(\mathbf{u}) = \min_{J(\mathbf{w})\geqslant\lambda_G} I(\mathbf{w}), \quad J(\mathbf{u}) \geqslant \lambda_G \tag{2.1.2}$$

are equivalent.

Proof. Let $\mathbf{u}$ be a solution of (2.1.2). If $J(\mathbf{u}) > \lambda_G$, there would exist $c < 1$ such that $J(c\mathbf{u}) = \lambda_G$ and $I(c\mathbf{u}) = c^2 I(\mathbf{u}) < I(\mathbf{u})$, contradicting the minimality of $\mathbf{u}$. Hence $\mathbf{u}$ is a solution of (Min).

Now let $\mathbf{u}$ be a solution of (Min). If there existed $\mathbf{w}$ with $J(\mathbf{w}) \geqslant \lambda_G$ and $I(\mathbf{w}) < I(\mathbf{u})$, then, for some $c \leqslant 1$, $J(cW) = \lambda_G$ and, from the minimality of $\mathbf{u}$, $I(\mathbf{u}) \leqslant I(c\mathbf{w}) \leqslant I(\mathbf{w}) < I(\mathbf{u})$, which is absurd. □

Lemma 2.1.3. *Suppose that there exists a solution of the problem* (Min). *Then G is the set of solutions for* (Min).

Proof. Let $\mathbf{u}$ be a solution of (Min). Then, for some Lagrange multiplier $\mu \in \mathbb{R}$ and any $\mathbf{h} = (h_1, ..., h_M) \in (H^1(\mathbb{R}^d))^M$,

$$\langle -\Delta u_m + \omega u_m, h_m \rangle_{H^{-1}\times H^1} = \mu(p+1)\langle \sum_{j=1}^M k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m, h_m\rangle_{H^{-1}\times H^1},\ 1 \leqslant m \leqslant M.$$

Taking $\mathbf{h} = \mathbf{u}$,

$$\lambda_G^{\frac{1}{p+1}} I^1 = I^{\lambda_G} = I(\mathbf{u}) = \mu(p+1)J(\mathbf{u}) = \mu(p+1)\lambda_G$$

The definition of λ_G implies that $\mu(p+1) = 1$ and so $\mathbf{u} \in A$. Therefore

$$I(\mathbf{u}) = \lambda_G \text{ and } S(\mathbf{u}) = \left(\frac{1}{2} - \frac{1}{2p+2} \right) \lambda_G.$$

Now take $\mathbf{w} \in A$. We want to see that $S(\mathbf{w}) \geqslant S(\mathbf{u})$. Let $\gamma = J(\mathbf{w})$. Then

$$I(\mathbf{w}) = \gamma \text{ and } S(\mathbf{w}) = \left(\frac{1}{2} - \frac{1}{2p+2} \right) \gamma.$$

Set $\mathbf{x} = \left(\frac{\lambda_G}{\gamma}\right)^{\frac{1}{2p+2}} \mathbf{w}$. Then $J(\mathbf{x}) = \lambda_G$. Since $\mathbf{u}$ is a solution of (Min),

$$\lambda_G^{\frac{1}{p+1}} I^1 = I(\mathbf{u}) \leqslant I(\mathbf{x}) = \left(\frac{\lambda_G}{\gamma}\right)^{\frac{1}{p+1}} I(\mathbf{x}) = \left(\frac{\lambda_G}{\gamma}\right)^{\frac{1}{p+1}} \gamma$$

and so $\gamma \geqslant (I^1)^{\frac{p+1}{p}} = \lambda_G$. Hence

$$S(\mathbf{w}) = \left(\frac{1}{2} - \frac{1}{2p+2}\right)\gamma \geqslant \left(\frac{1}{2} - \frac{1}{2p+2}\right)\lambda_G = S(\mathbf{u}),$$

which implies $\mathbf{u} \in G$. If $\mathbf{w} \in G$, one must have equality in the above inequality. Then $J(\mathbf{w}) = \lambda_G$ and, since $\mathbf{u}, \mathbf{w} \in A$, $I(\mathbf{w}) = J(\mathbf{w}) = J(\mathbf{u}) = I(\mathbf{u})$. Therefore $\mathbf{w}$ is a solution of (Min). $\square$

The proof of Theorem 2.1.1 relies on an adaptation of the concentration-compactness principle of P.L.Lions ([49], [50]) to ensure that the weak limit of a minimizing sequence is still in the set of admissible functions. As it is known, the Sobolev injection $H^1(\Omega) \hookrightarrow L^{2p+2}(\Omega)$ is compact if Ω is a bounded set[1] (see Appendix A). The concentration-compactness principle describes the lack of compactness of the injection $H^1(\mathbb{R}^d) \hookrightarrow L^{2p+2}(\mathbb{R}^d)$, that is, it tells us what may happen to a bounded H^1 sequence, even if it does not have a strong sublimit in L^{2p+2}. One observes one of three possibilities: the sequence vanishes in every ball (*vanishing case*); the sequence *concentrates* on some fixed ball (*compactness case*); the sequence splits in a concentrated part and an escaping part (*dichotomy case*). Applying this principle to a minimizing sequence, one may actually exclude both the vanishing and dichotomy cases and conclude, using the compactness of the Sobolev injection on bounded sets, the existence of a strong sublimit in L^{2p+2}. This process is exemplified, for example, in [50, Theorem I.2], where this technique is used to prove the existence of solutions of (Min) in the scalar case $M = 1$. For the general case, we need two lemmata from [49], which we present without proof.

Lemma 2.1.4 (Concentration-compactness principle for $(H^1(\mathbb{R}^d))^M$). *Let $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ be a sequence in $(H^1(\mathbb{R}^d))^M$ such that*

$$\sum_{m=1}^{M} \omega \int |(u_n)_m|^2 dx = c.$$

Then, up to a subsequence, one has the one of the following:

1. *Compactness: there exist $y_n \in \mathbb{R}^d$ such that*

$$\forall \epsilon > 0\ \exists R > 0\ \sum_{m=1}^{M} \omega \int_{y_n + B_R} |(\mathbf{u}_n)_m|^2 dx \geqslant c - \epsilon;$$

2. *Evanescence:*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^d} \sum_{m=1}^{M} \int_{y+B_R} |(\mathbf{u}_n)_m|^2 dx = 0, \forall R > 0;$$

3. *Dichotomy: there exists $\alpha \in]0, c[$ such that, for any given $\epsilon > 0$, there are sequences $\{\mathbf{u}_n^1\}_{n\in\mathbb{N}}$ and $\{\mathbf{u}_n^2\}_{n\in\mathbb{N}}$ verifying*

[1] Actually, this simplifies considerably the proof of existence of ground-states for bounded domains.

- $|(\mathbf{u}_n)_m| \geqslant |(\mathbf{u}_n^1)_m| + |(\mathbf{u}_n^2)_m|$, $m = 1, \ldots, M$;
- $M(\mathbf{u}_n - (\mathbf{u}_n^1 + \mathbf{u}_n^2)) \leqslant \epsilon$;
- $\left|M(\mathbf{u}_n^1) - \alpha\right| \leqslant \epsilon$ *and* $\left|M(\mathbf{u}_n^2) - (c - \alpha)\right| \leqslant \epsilon$;
- $\sum_{m=1}^M \int |\nabla(\mathbf{u}_n)_m|^2 \geqslant \sum_{m=1}^M \int |\nabla(\mathbf{u}_n^1)_m|^2 + \sum_{m=1}^M \int |\nabla(\mathbf{u}_n^2)_m|^2 - \epsilon$;
- $\exists R > 0\ \exists y_n \in \mathbb{R}^d$: $\mathrm{supp}\ \mathbf{u}_n^1 \subset y_n + B_R$;
- $\mathrm{dist}(\mathrm{supp}\ \mathbf{u}_n^1, \mathrm{supp}\ \mathbf{u}_n^2) \to \infty$ *when* $n \to \infty$;
- $\sup_{y \in \mathbb{R}^d} \sum_{m=1}^M \omega \int_{y+B_R} |(\mathbf{u}_n^2)_m|^2 dx \leqslant M(\mathbf{u}_n^1) + \epsilon$.

Lemma 2.1.5 (Elimination of the evanescence case)**.** *Let* $1 < r \leqslant \infty$, $1 \leqslant q < \infty$, *with* $q \neq r^* := rN/(N-r)$ *if* $r < N$. *Suppose that* $\{u_n\}_{n \in \mathbb{N}}$ *is bounded in* $L^q(\mathbb{R}^d)$, $\{\nabla u_n\}_{n \in \mathbb{N}}$ *is bounded in* $L^r(\mathbb{R}^d)$ *and that*

$$\sup_{y \in \mathbb{R}^d} \int_{y+B_R} |u_n|^q dx \to 0, \text{ for some } R > 0.$$

Then $u_n \to 0$ *in* $L^s(\mathbb{R}^d)$, *for any* s *between* q *and* r^*.

Proof of Theorem 2.1.1: By Lemma 2.1.3, it suffices to prove that (Min) has a solution. Let $\{\mathbf{u}_n\}$ be a minimizing sequence of (Min). From Remark 2.1.2, it follows that

$$\inf M(\mathbf{u}_n) > 0.$$

Up to a subsequence, there exists $c > 0$ such that $M(\mathbf{u}_n) \to c$. Define

$$\mathbf{w}_n = \left(\frac{c}{M(\mathbf{u}_n)}\right)^{1/2} \mathbf{u}_n \ \text{ so that } \ M(\mathbf{w}_n) = c.$$

We apply the concentration-compactness principle (cf. Lemma 2.1.4) to the sequence $\{\mathbf{w}_n\}_{n \in \mathbb{N}}$. Now we exclude the evanescence and dichotomy cases.

- In the evanescence case, it follows from Lemma 2.1.5 that $\mathbf{w}_n \to 0$ in $(L^{2p+2}(\mathbb{R}^d))^M$, contradicting
$$\lambda_G = \lim J(\mathbf{w}_n) \lesssim \lim \sum_{m=1}^M \|(\mathbf{w}_n)_m\|_{2p+2}^{2p+2} = 0.$$

- In the dichotomy case, fixed $\epsilon > 0$, we obtain a decomposition of $\mathbf{w}_n$ in $\mathbf{w}_n^1$ and $\mathbf{w}_n^2$ as in Lemma 2.1.4. Up to a subsequence, we have
$$\lim J(\mathbf{w}_n^1) = \lambda_1^\epsilon, \quad \lim J(\mathbf{w}_n^2) = \lambda_2^\epsilon.$$
and so $|\lambda_G - \lambda_1^\epsilon - \lambda_2^\epsilon| \leqslant \epsilon$. Up to a subsequence, define
$$\lambda_1 = \lim_{\epsilon \to 0} \lambda_1^\epsilon, \quad \lambda_2 = \lim_{\epsilon \to 0} \lambda_2^\epsilon.$$

Notice that $\lambda = \lambda_1 + \lambda_2$. Without loss of generality, assume that $\lambda_1 \leqslant \lambda_2$. If $\lambda_1 \leqslant 0$, then

$$\lambda_G - \epsilon + \lambda_1^\epsilon < \lambda_2^\epsilon$$

and

$$\begin{aligned} I^{\lambda_G} &= \lim I(\mathbf{u}_n) = \lim I(\mathbf{w}_n) \geqslant \liminf I(\mathbf{w}_n^1) + I(\mathbf{w}_n^2) - \epsilon \\ &\geqslant \liminf \alpha + I\left(\left(\frac{\lambda_2}{J(\mathbf{w}_n^2)}\right)^{\frac{1}{2p+2}} \mathbf{w}_n^2\right) - 2\epsilon \geqslant \alpha + I^{\lambda_2} - 2\epsilon. \end{aligned}$$

Taking the limit $\epsilon \to 0$, we obtain

$$\lambda_G \leqslant \lambda_2, \quad \lambda_G^{\frac{1}{p+1}} I^1 = I^{\lambda_G} \geqslant \alpha + I^{\lambda_2} > \lambda_2^{\frac{1}{p+1}} I^1,$$

which is absurd. Hence $\lambda_1, \lambda_2 > 0$ and so

$$\begin{aligned} I^{\lambda_G} &= \lim I(\mathbf{u}_n) = \lim I(\mathbf{w}_n) \geqslant \liminf I(\mathbf{w}_n^1) + I(\mathbf{w}_n^1) - \epsilon \\ &\geqslant \liminf I\left(\left(\frac{\lambda_1^\epsilon}{J(\mathbf{w}_n^1)}\right)^{\frac{1}{2p+2}} \mathbf{w}_n^1\right) + I\left(\left(\frac{\lambda_2^\epsilon}{J(\mathbf{w}_n^2)}\right)^{\frac{1}{2p+2}} \mathbf{w}_n^2\right) - \epsilon \\ &\geqslant I^{\lambda_1^\epsilon} + I^{\lambda_2^\epsilon} - \epsilon. \end{aligned}$$

Passing to the limit $\epsilon \to 0$,

$$\lambda_G^{\frac{1}{p+1}} I^1 = I^{\lambda_G} \geqslant I^{\lambda_1} + I^{\lambda_2} = \lambda_1^{\frac{1}{p+1}} I^1 + (\lambda_G - \lambda_1)^{\frac{1}{p+1}} I^1,$$

which is impossible, since the mapping $\lambda \mapsto \lambda^{\frac{1}{p+1}} I_1$ is concave.

Therefore $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ satisfies the compactness alternative. Since $M(\mathbf{u}_n) \to c$, it follows from the definition of $\mathbf{w}_n$ that $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ also satisfies the compactness alternative, that is, there exist $y_n \in \mathbb{R}^d$ such that

$$\forall \epsilon > 0 \ \exists R_\epsilon : \sum_{m=1}^{M} \int_{y_n + B_{R_\epsilon}} |(\mathbf{u}_n)_m|^2 dx \geqslant c - \epsilon.$$

Since $\mathbf{u}_n(\cdot - y_n)$ is bounded in $(H^1(\mathbb{R}^d))^M$, up to a subsequence, we have $\mathbf{u}_n(\cdot - y_n) \rightharpoonup \mathbf{u}$ in $H^1(\mathbb{R}^d)$. For each $\epsilon > 0$ fixed, it follows from the compact Sobolev injection $H^1(B_R) \hookrightarrow L^2(B_R)$ that $\mathbf{u}_n(\cdot - y_n) \to \mathbf{u}$ in $(L^2(B_R))^M$.

For any $R > 0$ and $u \in H^1(\mathbb{R}^d)$, if $\phi \in W^{1,\infty}(\mathbb{R}^d)$ is such that $\phi \equiv 1$ in B_R and $\phi \equiv 0$ in $\mathbb{R}^d \backslash B_{2R}$, then

$$\|u\|_{L^{2p+2}(B_R)}^{2p+2} \leqslant \|\phi u\|_{L^{2p+2}(B_R)}^{2p+2} \lesssim \|u\|_{L^2(B_{2R})}^{2p+2-dp} \|\nabla(\phi u)\|_2^{dp}$$

and

$$\|u\|_{L^{2p+2}(\mathbb{R}^d \backslash B_{2R})}^{2p+2} \leqslant \|(1-\phi)u\|_{L^{2p+2}(\mathbb{R}^d \backslash B_R)}^{2p+2} \lesssim \|u\|_{L^2(B_R)}^{2p+2-dp} \|\nabla(\phi u)\|_2^{dp}.$$

It then follows from the compactness alternative that

$$\begin{aligned}|J(\mathbf{u})-J(\mathbf{u_n})| &\lesssim \sum_{m=1}^{M}\|\mathbf{u}_m-(\mathbf{u}_n)_m(\cdot-y_n)\|_{L^{2p+2}(B_{R_\epsilon})}^{2p+2}\\ &\quad+\|\mathbf{u}_m\|_{L^{2p+2}(\mathbb{R}^d\setminus B_{R_\epsilon})}^{2p+2}+\|(\mathbf{u}_n)_m\|_{L^{2p+2}(\mathbb{R}^d\setminus B_{R_\epsilon})}^{2p+2}\\ &\lesssim \sum_{m=1}^{M}\|\mathbf{u}_m-(\mathbf{u}_n)_m(\cdot-y_n)\|_{L^{2}(B_{2R_\epsilon})}^{2p+2-dp}+\\ &\quad+\|\mathbf{u}_m\|_{L^{2}(\mathbb{R}^d\setminus B_{R_\epsilon/2})}^{2p+2-dp}+\|(\mathbf{u}_n)_m\|_{L^{2}(\mathbb{R}^d\setminus B_{R_\epsilon/2})}^{2p+2-dp}\lesssim\epsilon,\end{aligned}$$

for n and R_ϵ large. Hence, for any ϵ, $|J(\mathbf{u})-\lambda_G|\lesssim\epsilon$, and so $J(\mathbf{u})=\lambda_G$. Finally, from the weak semi-continuity of the H^1 norm,

$$I(\mathbf{u})\leqslant \liminf I(\mathbf{u}(\cdot-y_n))=\liminf I(\mathbf{u}_n)=I^{\lambda_G}\leqslant I(\mathbf{u}).$$

Thus $\mathbf{u}$ is a solution of (Min), as we wanted. □

Remark 2.1.3. The exact same proof can be carried out for system (M-NLS') and so the question of existence of ground-states is settled for the rest of Part I.

For the case where all the components attract each other, one may improve the above result using Schwarz symmetrization. This fact is not new (see [47]), however we display the following result for the sake of completeness.

Proposition 2.1.6. *If $k_{jm}\geqslant 0,\ 1\leqslant m\neq j\leqslant M$, then* (Min) *has a positive, radial, decreasing solution.*

Proof. Let $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of (Min). Defining $|\mathbf{w}|:=(|w_1|,...,|w_M|)$, clearly $\{|\mathbf{u}_n|\}_{n\in\mathbb{N}}$ is also a minimizing sequence. Let $|\mathbf{w}|^*=(|w_1|^*,...,|w_M|^*)$ be the vector of the Schwarz symmetrizations of the components of $|\mathbf{w}|$. The properties of the symmetrization imply that $\{|\mathbf{u}_n|^*\}_{n\in\mathbb{N}}$ satisfies

$$J(|\mathbf{u}_n|^*)\geqslant\lambda_G,\quad I^{\lambda_G}\leqslant\liminf I(|\mathbf{u}_n|^*)\leqslant\lim I(\mathbf{u}_n)=I^{\lambda_G}.$$

Using a compactness result for Schwarz symmetrizations, up to a subsequence, we have

$$|\mathbf{u}_n|^*\rightharpoonup\mathbf{u}\text{ in }(H^1(\mathbb{R}^d))^M,\quad |\mathbf{u}_n|^*\to\mathbf{u}\text{ in }(L^2(\mathbb{R}^d)\cap L^{2p+2}(\mathbb{R}^d))^M.$$

Hence

$$J(\mathbf{u})=\lim J(|\mathbf{u}_n|^*)\geqslant\lambda_G,\qquad I^{\lambda_G}\leqslant I(\mathbf{u})\leqslant\liminf I(|\mathbf{u}_n|^*)=I^{\lambda_G}.$$

Therefore $\mathbf{u}$ is a solution of (2.1.2) and, by Lemma 2.1.2, it is a solution of (Min). □

The next proposition gives another minimization problem for which the set of solutions is G. This characterization will be especially useful in Section 3.3 to study the nontriviality of incoherent ground-states.

Proposition 2.1.7. *Define the Nehari manifold as*

$$\mathcal{N} = \{\mathbf{u} \in (H^1(\mathbb{R}^d))^M : \ \mathbf{u} \not\equiv 0, \ I(\mathbf{u}) = J(\mathbf{u})\}.$$

Then the minimization problems (Min) *and*

$$I(\mathbf{u}) = \min_{\mathbf{w}\in\mathcal{N}} I(\mathbf{w}), \quad \mathbf{u} \in \mathcal{N} \tag{2.1.3}$$

are equivalent.

Proof. Let $\mathbf{u}$ be a solution of (Min). Then $\mathbf{u} \in G \subset \mathcal{N}$ and $\mathbf{w} \in \mathcal{N}$. Since $J(\mathbf{w}) = I(\mathbf{w}) > 0$, one may define $\lambda > 0$ so that $J(\lambda\mathbf{w}) = J(\mathbf{u})$, that is,

$$\lambda = \left(\frac{J(\mathbf{u})}{J(\mathbf{w})}\right)^{\frac{1}{2p+2}}.$$

Then, from the fact that $\mathbf{u}$ minimizes (Min),

$$\left(\frac{J(\mathbf{u})}{J(\mathbf{w})}\right)^{\frac{1}{p+1}} J(\mathbf{w}) = \left(\frac{J(\mathbf{u})}{J(\mathbf{w})}\right)^{\frac{1}{p+1}} I(\mathbf{w}) = I(\lambda\mathbf{w}) \geqslant I(\mathbf{u}) = J(\mathbf{u}).$$

This implies that $J(\mathbf{w}) \geqslant J(\mathbf{u})$ and, from Lemma 2.1.2, $I(\mathbf{w}) \geqslant I(\mathbf{u})$. Since $u \in \mathcal{N}$, this implies that $\mathbf{u}$ is a solution of (2.1.3). On the other hand, if $\mathbf{w}$ is a solution of (2.1.3), then $J(\mathbf{w}) = I(\mathbf{w}) = I(\mathbf{u}) = J(\mathbf{u})$ and so $\mathbf{w}$ is a solution of (Min). □

2.2 Ground-states: characterization results

This section will focus on three characterization results, each of independent interest: the first considers an all-attractive setting (that is, $k_{jm} \geqslant 0$ for $m \neq j$); the second focuses on what happens when a group of components repels the remaining ones (*i.e.*, for some $1 \leqslant L < M$, $k_{jm} < 0$ for $m \leqslant L$ and $j > L$); the third takes the general setting.

We start with an abstract result which relates solutions of minimization problems for vector-valued functions with the solutions of some related problem for scalar functions. Given a real vector space X, consider operators $I_1, J_1 : X \to \mathbb{R}$ and $C : X \times X \to \mathbb{R}$ such that

(H1) I_1 is homogeneous of degree $\alpha > 0$;

(H2) J_1 is homogeneous of degree $2\beta > 0$ and $J_1(w) > 0$ if $w \neq 0$;

(H3) $C(\eta w, \xi w) = \eta^\beta \xi^\beta J_1(w)$ and $C(w,z) \leqslant J_1(w)^{1/2} J_1(z)^{1/2}$, $\forall\ w, z \in X\ \forall\ \eta, \xi > 0$.

Given $c_{jm} \in \mathbb{R}$, $1 \leqslant m, j \leqslant M$, with $c_{jm} \geqslant 0$ if $m \neq j$, we define

$$I(\mathbf{u}) := \sum_{m=1}^{M} I_1(u_m) \text{ and } J(\mathbf{u}) := \sum_{j,m=1}^{M} c_{jm} C(u_m, u_j).$$

Lemma 2.2.1. *Fix $\gamma > 0$. Suppose that the family $\mathcal{M} \subset X$ of solutions of the minimization problem*

$$I_1(u) = \min_{J_1(w)=\gamma} I_1(w), \quad J_1(u) = \gamma$$

is nonempty and that $\mathbf{u} = (u_1, ..., u_M) \in (X\backslash\{0\})^M$ is a solution of the minimization problem

$$I(\mathbf{u}) = \min_{J(\mathbf{w})\geqslant J(\mathbf{u})} I(\mathbf{w}).$$

Then there exist $d_m > 0$ and $P_m \in \mathcal{M}$ such that $\mathbf{u} = (d_m P_m)_{1\leqslant m\leqslant M}$.

Proof. Let $R \in \mathcal{M}$. First of all, we have

$$J_1\left(\left(\frac{J_1(R)}{J_1(u_m)}\right)^{\frac{1}{2\beta}} u_m\right) = J_1(R),\ 1 \leqslant m \leqslant M.$$

Suppose, by absurd, and without loss of generality, that $d_1u_1 \notin \mathcal{M}, \forall d_1 > 0$. By the minimality of $\mathcal{M}$,

$$I_1\left(\left(\frac{J_1(R)}{J_1(u_1)}\right)^{\frac{1}{2\beta}} u_1\right) > I_1(R)$$

and

$$I_1\left(\left(\frac{J_1(R)}{J_1(u_m)}\right)^{\frac{1}{2\beta}} u_m\right) \geqslant I_1(R),\ 2 \leqslant m \leqslant M.$$

This implies that

$$\begin{aligned} I(\mathbf{u}) = I_1(u_1) + \sum_{m=2}^{M} I_1(u_m) &> I_1\left(\left(\frac{J_1(u_1)}{J_1(R)}\right)^{\frac{1}{2\beta}} R\right) + \sum_{m=2}^{M} I_1\left(\left(\frac{J_1(u_m)}{J_1(R)}\right)^{\frac{1}{2\beta}} R\right) \\ &= I\left(\left(\frac{J_1(u_1)}{J_1(R)}\right)^{\frac{1}{2\beta}} R, ..., \left(\frac{J_1(u_M)}{J_1(R)}\right)^{\frac{1}{2\beta}} R\right). \end{aligned}$$

By the minimality of $\mathbf{u}$,

$$J\left(\left(\frac{J_1(u_1)}{J_1(R)}\right)^{\frac{1}{2\beta}} R, ..., \left(\frac{J_1(u_M)}{J_1(R)}\right)^{\frac{1}{2\beta}} R\right) < J(\mathbf{u}).$$

Using the definition of J,

$$\begin{aligned} \sum_{j,m=1,m\neq j}^{M} c_{jm} C\left(\left(\frac{J_1(u_m)}{J_1(R)}\right)^{\frac{1}{2\beta}} R, \left(\frac{J_1(u_j)}{J_1(R)}\right)^{\frac{1}{2\beta}} R\right) &< \sum_{j,m=1,m\neq j}^{M} c_{jm} C(u_m, u_j) \\ &\leqslant \sum_{j,m=1,m\neq j}^{M} c_{jm} J_1(u_m)^{\frac{1}{2}} J_1(u_j)^{\frac{1}{2}}. \end{aligned}$$

However, by the homogeneity of C,

$$\sum_{j,m=1,m\neq j}^{M} c_{jm} C\left(\left(\frac{J_1(u_m)}{J_1(R)}\right)^{\frac{1}{2\beta}} R, \left(\frac{J_1(u_j)}{J_1(R)}\right)^{\frac{1}{2\beta}} R\right) = \sum_{j,m=1,m\neq j}^{M} c_{jm} J_1(u_m)^{\frac{1}{2}} J_1(u_j)^{\frac{1}{2}},$$

which is absurd. $\square$

We now apply this general principle to the matter at hand:

Theorem 2.2.2. *Suppose* (PC) *and that* $k_{jm} \geqslant 0$, $\forall m \neq j$. *Then* $\mathbf{u} \in G^*$ *if and only if there exist* $\theta_m \in \mathbb{R}$, $m = 1, ..., M$, *and* $y \in \mathbb{R}^d$ *such that*

$$\mathbf{u} = (a_m e^{i\theta_m} Q(\cdot + y))_{1 \leqslant m \leqslant M} \tag{2.2.1}$$

where

$$(a_1, ..., a_M) \in S^+ = \left\{ \mathbf{b} = (b_1, ..., b_M) \in (\mathbb{R}^+)^M : \sum_{j=1}^{M} k_{jm} b_m^{p-1} b_j^{p+1} = 1, m = 1, ..., M \right\}$$

and

$$\sum_{m=1}^{M} a_m^2 I_1(Q) = \min \left\{ \min_{\mathbf{b} \in S^+} \left\{ \sum_{m=1}^{M} b_m^2 I_1(Q) \right\}, \min_{\mathbf{w} \in G \setminus G^*} I(\mathbf{w}) \right\}. \tag{2.2.2}$$

Remark 2.2.1. Even though the result only characterize, *a priori*, the elements of G^*, one may obtain the description of G. Simply notice that, if $\mathbf{u} \in G \setminus G^*$, then $\mathbf{u}$ has L nonzero components, with $1 \leqslant L < M$. If $\mathbf{u}^*$ is the vector formed by such components, it has to be a ground-state of a (L-NLS) system. By Theorem 2.2.2 applied with $M = L$, we find the explicit expression of $\mathbf{u}^*$ and therefore of $\mathbf{u}$.

Proof. We divide the proof in three steps:
Step 1: $\mathbf{u} \in G^*$ satisfies (2.2.1), with $\mathbf{a} = (a_1, ..., a_M) \in S^+$.

Let $\mathbf{u} \in G^*$. By Lemmata 1.2.2, 1.2.3, 2.1.2 and 2.1.3, we may apply Lemma 2.2.1 with

$$I_1(u) = \int |\nabla u|^2 + \omega |u|^2, \quad J_1(u) = \int |u|^{2p+2}, \quad C(u, v) = \int |u|^{p+1} |v|^{p+1}$$

and conclude that there exist, for each $1 \leqslant m \leqslant M$, $a_m > 0$, $\theta_m \in \mathbb{R}$ and $y_m \in \mathbb{R}^d$ such that

$$\mathbf{u} = (a_m e^{i\theta_m} Q(\cdot + y_m))_{1 \leqslant m \leqslant M}.$$

If there exist m_0, j_0 such that $y_{m_0} \neq y_{j_0}$, one easily sees that there exists $D \subset \mathbb{R}^d$ of positive measure such that, for all $x \in D$, $Q(x + y_{m_0}) \neq Q(x + y_{j_0})$ and so, using Young's inequality,

$$Q(x + y_{m_0})^{p+1} Q(x + y_{j_0})^{p+1} < \frac{1}{2} Q(x + y_{m_0})^{2p+2} + \frac{1}{2} Q(x + y_{j_0})^{2p+2}, \quad x \in D.$$

On the other hand, we have in general

$$Q(x+y_m)^{p+1}Q(x+y_j)^{p+1} \leqslant \frac{1}{2}Q(x+y_m)^{2p+2}+\frac{1}{2}Q(x+y_j)^{2p+2}, \quad x \in \mathbb{R}^d,\ 1 \leqslant m, j \leqslant M.$$

Consequently,

$$\begin{aligned}
&\int (a_m Q(\cdot + y_m))^{p+1}(a_j Q(\cdot + y_j))^{p+1} \\
&\leqslant a_m^{p+1}a_j^{p+1}\left(\frac{1}{2}\int Q(\cdot + y_m)^{2p+2} + \frac{1}{2}\int Q(\cdot + y_j)^{2p+2}\right) \\
&= a_m^{p+1}a_j^{p+1}\int Q^{2p+2} = \int (a_m Q)^{p+1}(a_j Q)^{p+1},
\end{aligned}$$

with strict inequality if $m = m_0$ and $j = j_0$. Therefore,

$$\lambda_G = J(\mathbf{u}) < J((a_m Q)_{1\leqslant m\leqslant M}) =: \lambda$$

. Hence

$$J\left(\left(\frac{\lambda_G}{\lambda}\right)^{\frac{1}{2p+2}}(a_m Q)_{1\leqslant m\leqslant M}\right) = \lambda_G$$

and

$$I\left(\left(\frac{\lambda_G}{\lambda}\right)^{\frac{1}{2p+2}}(a_m Q)_{1\leqslant m\leqslant M}\right) < I\left((a_m Q)_{1\leqslant m\leqslant M}\right) = I(\mathbf{u}),$$

which contradicts the minimality of $\mathbf{u}$. Therefore $y_m = y_j$, for any $1 \leqslant m, j \leqslant M$ and so $\mathbf{u}$ is of the form (2.2.1).

Replacing the formula of $\mathbf{u}$ into the system (M-NLS), we derive

$$\sum_{j=1}^{M} k_{jm} a_m^{p-1} a_j^{p+1} = 1, \quad \forall 1 \leqslant m \leqslant M.$$

Hence $\mathbf{a} \in S^+$.

Step 2: If $\mathbf{u}$ is of the form (2.2.1), with $\mathbf{a} \in S^+$, $\mathbf{u} \in A$.

Simply notice that $\mathbf{u}$ satisfies the system (M-NLS), using the conditions of S^+.

Step 3: Conclusion.

Let $\mathbf{u} \in G^*$. If $\mathbf{a}$ does not satisfy (2.2.2), then either

$$\min_{\mathbf{w}\in G\backslash G*} I(\mathbf{w}) < \sum_{m=1}^{M} a_m^2 I_1(Q) = I(\mathbf{u})$$

or there exists $\mathbf{b} \in S^+$ such that

$$\sum_{m=1}^{M} b_m^2 I_1(Q) < \sum_{m=1}^{M} a_m^2 I_1(Q).$$

In the first case, there would exist $\mathbf{w} \in G\backslash G^*$ with $I(\mathbf{w}) < I(\mathbf{u})$, which contradicts $\mathbf{u} \in G$. In the second case, given $\theta_m \in \mathbb{R}$, $1 \leqslant m \leqslant M$, and $y \in \mathbb{R}^d$,

$$\mathbf{w} := (b_m e^{i\theta_m} Q(\cdot + y))_{1 \leqslant m \leqslant M}$$

is in A. Moreover,

$$\begin{aligned}
S(\mathbf{w}) &= \left(\frac{1}{2} - \frac{1}{2p+2}\right) I(\mathbf{w}) = \left(\frac{1}{2} - \frac{1}{2p+2}\right) \sum_{m=1}^{M} b_m^2 I_1(Q) \\
&< \left(\frac{1}{2} - \frac{1}{2p+2}\right) \sum_{m=1}^{M} a_m^2 I_1(Q) = S(\mathbf{u}),
\end{aligned}$$

which contradicts $\mathbf{u} \in G$. We conclude that $\mathbf{a}$ satisfies (2.2.2). It remains to prove that $\mathbf{w} \in G$. In fact,

$$\begin{aligned}
S(\mathbf{w}) &= \left(\frac{1}{2} - \frac{1}{2p+2}\right) I(\mathbf{w}) = \left(\frac{1}{2} - \frac{1}{2p+2}\right) \sum_{m=1}^{M} b_m^2 I_1(Q) \\
&= \left(\frac{1}{2} - \frac{1}{2p+2}\right) \sum_{m=1}^{M} a_m^2 I_1(Q) = S(\mathbf{u}).
\end{aligned}$$

Therefore $\mathbf{w} \in G$, which ends the proof. $\square$

Theorem 2.2.3. *Suppose* (PC) *and that there exists a partition* $\{Y_k\}_{1 \leqslant k \leqslant K}$ *of* $\{1, ..., M\}$ *such that, given* $1 \leqslant m \neq j \leqslant M$,

$$k_{jm} \geqslant 0 \textit{ if and only if } \exists k : j, m \in Y_k.$$

Then, if $\mathbf{u} = (u_1, ..., u_M) \in G$, *there exists* $k \in \{1, ..., K\}$ *such that* $u_m = 0, \forall m \notin Y_k$.

Remark 2.2.2. In the conditions of Theorem 2.2.3, we can also characterize the set G, since the vector of the nonzero components of a given ground-state of (M-NLS) is a ground-state for a (L-NLS) system, with $L < M$, where all the nondiagonal coupling coefficients are nonnegative. Therefore it is possible to apply Theorem 2.2.2 to (L-NLS), and thus obtain the description of the initial ground-state.

Proof. Step 1. The partition $\{Y_k\}_{1 \leqslant k \leqslant K}$ defines an equivalence relation in the set $\{1, ..., M\}$:

$$m \sim j \text{ if and only if } \exists k \; j, m \in Y_k.$$

We claim that (Min) is equivalent to

$$\sum_{m=1}^{M} I(u_m) = \min_{B} \sum_{m=1}^{M} I(w_m), \quad (u_1, ..., u_M) \in B \tag{2.2.3}$$

where

$$B = \Big\{(w_1, ..., w_M) \in (H^1(\mathbb{R}^d))^M : J(w_1, \ldots, w_M) = \lambda_G,$$
$$C(w_m, w_j) = 0 \text{ if } m \nsim j\Big\}.$$

To see this, suppose that $\mathbf{u}$ is a solution of (Min). If $C(u_m, u_j) = 0, \forall m \nsim j$, then $\mathbf{u}$ is a solution of (2.2.3). By absurd, suppose that there exist $m_0 \nsim j_0$ such that $C(u_{m_0}, u_{j_0}) \neq 0$. Let $\mathbf{u}^R$ be defined by

$$u_m^R = u_m, \text{ if } m \nsim j_0, \quad u_m^R = u_m(\cdot + Re_1) \text{ if } m \sim j_0.$$

Then, for large R, $C(u_m^R, u_j^R) \leqslant C(u_m, u_j)$ if $m \nsim j$ (with strict inequality if $m = m_0$, $j = j_0$) and $C(u_m^R, u_j^R) = C(u_m, u_j)$ if $m \sim j$. Hence, $J(\mathbf{u}^R) > J(\mathbf{u})$. However, from the expression of $\mathbf{u}^R$, $I(\mathbf{u}^R) = I(\mathbf{u})$. This is absurd, by Lemma 2.1.2. Hence $C(u_m, u_j) = 0, \forall m \nsim j$ and $\mathbf{u}$ is a solution of (2.2.3).

On the other hand, if $\mathbf{u}$ is a solution of (2.2.3), suppose, by contradiction, that it is not a solution of (Min). This implies that $I(\mathbf{u}) > I(\mathbf{z})$, where $\mathbf{z}$ is a solution on (Min) (whose existence is ensured by Theorem 2.1.1). However, by the above, $\mathbf{z}$ is a solution of (2.2.3), meaning that $I(\mathbf{z}) = I(\mathbf{u})$, which is absurd.

Step 2. Let

$$K_G = \{k \in \{1, ..., K\} : \exists \mathbf{u} \in (H^1(\mathbb{R}^d))^M : \sum_{j,m \in Y_k} k_{jm} C(\mathbf{u}_m, \mathbf{u}_j) > 0\}.$$

For $\mathbf{z} = (z_1, ..., z_M) \in B$, define

$$K^+ = \{k \in \{1, ..., K\} : \sum_{j,m \in Y_k} k_{jm} C(z_m, z_j) > 0\} \subset K_G,$$

and $\tilde{\mathbf{z}}$ as $\tilde{z}_m = z_m,\ m \in Y_k, k \in K^+$ and $\tilde{z}_m = 0,\ m \in Y_k, k \notin K^+$.

Then

$$\mathbf{w} := \left(\frac{J(\mathbf{z})}{J(\overline{\mathbf{z}})}\right)^{\frac{1}{2p+2}} \tilde{\mathbf{z}} \in B \text{ and } I\left(\left(\frac{J(\mathbf{z})}{J(\tilde{\mathbf{z}})}\right)^{\frac{1}{2p+2}} \tilde{\mathbf{z}}\right) \leqslant I(\tilde{\mathbf{z}}) \leqslant I(\mathbf{z}).$$

with strict inequality if $\mathbf{z} \neq \tilde{\mathbf{z}}$.

For each $1 \leqslant k \leqslant K_G$, let $\mathbf{u}^k \in H^1(\mathbb{R}^d)^{|Y_k|}$ be a ground-state of the system formed by the equations of the m-th components, with $m \in Y_k$. Fix $k \in K^+$. Then, defining

$$c_k = \left(\frac{\sum_{j,m \in Y_k} k_{jm} C(w_m, w_j)}{\sum_{j,m \in Y_k} k_{jm} C(u_m^k, u_j^k)}\right)^{\frac{1}{2p+2}}$$

we have

$$\sum_{j,m \in Y_k} k_{jm} C\left(\frac{w_m}{c_k}, \frac{w_j}{c_k}\right) = \sum_{j,m \in Y_k} k_{jm} C\left(u_m^k, u_j^k\right).$$

Since $\mathbf{u}^k$ is a solution of (Min), with $M = |Y_k|$ and $j, m \in Y_k$, we obtain

$$\sum_{m\in Y_k} I_1(u_m^k) \leqslant \sum_{m\in Y_k} I_1\left(\frac{w_m}{c_k}\right) = \frac{1}{c_k^2}\sum_{m\in Y_k} I_1(w_m)$$

and so

$$\sum_{k\in K^+}^{K} c_k^2 \sum_{m\in Y_k} I_1(u_m^k) \leqslant \sum_{m=1}^{M} I_1(w_m).$$

Let k_0 be such that

$$\frac{\left(\sum_{m\in k_0} I_1(u_m^{k_0})\right)^{p+1}}{\sum_{j,m\in Y_{k_0}} k_{jm} C(u_m^{k_0}, u_j^{k_0})} \leqslant \frac{\left(\sum_{m\in k} I_1(u_m^{k})\right)^{p+1}}{\sum_{j,m\in Y_{k}} k_{jm} C(u_m^{k}, u_j^{k})},\ \forall k \in K_G.$$

Let $\mathbf{u} \in (H^1(\mathbb{R}^d))^M$ be defined by $u_m = 0$, if $m \notin Y_{k_0}$, and, otherwise,

$$\begin{aligned} u_m &= \left(\frac{\lambda_G}{\sum_{j,m\in Y_{k_0}} k_{jm} C(u_m^{k_0}, u_j^{k_0})}\right)^{\frac{1}{2p+2}} u_m^{k_0} \\ &= \left(\frac{\sum_{k\in K^+}\sum_{j,m\in Y_k} k_{jm} C(w_m, w_j)}{\sum_{j,m\in Y_{k_0}} k_{jm} C(u_m^{k_0}, u_j^{k_0})}\right)^{\frac{1}{2p+2}} u_m^{k_0}. \end{aligned}$$

It is easy to see that $J(\mathbf{u}) = \lambda_G$ and, by the definition of k_0,

$$\begin{aligned} I(\mathbf{u}) &= \left(\sum_{k\in K^+}\sum_{j,m\in Y_k} k_{jm} C(u_m, u_j) \frac{\left(\sum_{m\in Y_{k_0}} I_1(u_m^{k_0})\right)^{p+1}}{\sum_{j,m\in Y_{k_0}} k_{jm} C(u_m^{k_0}, u_j^{k_0})}\right)^{\frac{1}{p+1}} \\ &\leqslant \left(\sum_{k\in K^+} \frac{\sum_{j,m\in Y_k} k_{jm} C(w_m, w_j)}{\sum_{j,m\in Y_k} k_{jm} C(u_m^k, u_j^k)} \left(\sum_{m\in Y_k} I_1(u_m^k)\right)^{p+1}\right)^{\frac{1}{p+1}} \\ &= \left(\sum_{k\in K^+}\left(c_k^2 \sum_{m\in Y_k} I_1(u_m^k)\right)^{p+1}\right)^{\frac{1}{p+1}} \leqslant \sum_{k\in K^+} c_k^2 \sum_{m\in Y_k} I_1(u_m^k) \\ &\leqslant \sum_{m=1}^{M} I_1(w_m) = I(\mathbf{w}) \leqslant I(\mathbf{z}). \end{aligned}$$

Therefore $\mathbf{u}$ is a solution of (2.2.3) and, by Lemma 2.1.3, $\mathbf{u} \in G$. Finally, if $\mathbf{z} \in G$, then $I(\mathbf{z}) = I(\mathbf{u})$, which implies that all of the above inequalities must be in fact equalities. From the above computation, we obtain, for some $1 \leqslant k_Z \leqslant K$, $c_k = 0$, $\forall k \neq k_Z$ and $\tilde{\mathbf{z}} = \mathbf{z}$. Hence $K^+ = \{k_Z\}$ and the proof is concluded. □

Theorem 2.2.4. *Consider system* (M-NLS) *and suppose* (PC). *Define* $f : (\mathbb{R}_0^+)^M \to \mathbb{R}$,

$$f(\mathbf{y}) = \sum_{j,m=1}^{M} k_{jm} y_m^{p+1} y_j^{p+1}$$

and let $\mathcal{Y} \subset (\mathbb{R}_0^+)^M$ *be the set of solutions of*

$$f(\mathbf{y}_0) = f_{max} := \max_{|\mathbf{y}|=1} f(\mathbf{y}), \quad |\mathbf{y}_0| = 1.$$

Then $\mathbf{u} \in G$ *if and only if there exist* $a_m \in \mathbb{C}$, $y \in \mathbb{R}^d$, $1 \leqslant m \leqslant M$, *such that* $(f_{max})^{1/2p}(|a_1|, ..., |a_M|) \in \mathcal{Y}$ *and*

$$\mathbf{u} = (a_m Q(\cdot + y))_{1 \leqslant m \leqslant M}, \quad Q \text{ ground-state of (NLS).} \tag{2.2.4}$$

In particular, $G^* \neq \emptyset$ *if and only if there exists* $\mathbf{y} \in \mathcal{Y}$ *such that* $y_m \neq 0$, $m = 1, ..., M$. *Moreover* $G = G^*$ *if and only if all elements of* $\mathcal{Y}$ *have no zero components.*

Proof. Take $\mathbf{u} \in G$. Define $\hat{\mathbf{u}}(x) = (|u_1(x)|, ..., |u_M(x)|)$ and $u(x) = |\hat{\mathbf{u}}(x)|$. Since $J(\mathbf{u}) = J(\hat{\mathbf{u}})$ and $I(\mathbf{u}) \geqslant I(\hat{\mathbf{u}})$, $\hat{\mathbf{u}}$ is a minimizer. Fix $\mathbf{y} \in \mathcal{Y}$. Now notice that

$$\begin{aligned} J(\hat{\mathbf{u}}) &= \int f(\hat{\mathbf{u}}(x))dx = \int f\left(\frac{\hat{\mathbf{u}}(x)}{u(x)}\right) u(x)^{2p+2}dx \leqslant \int f(\mathbf{y})u(x)^{2p+2}dx \\ &= \int f\left(u(x)\mathbf{y}\right) dx = J(u\mathbf{y}) \end{aligned}$$

and that, from Cauchy-Schwarz inequality,

$$\begin{aligned} I(u\mathbf{y}) &= \int u(x)^2|\mathbf{y}|^2 + |\nabla(u(x))|^2 |\mathbf{y}|^2 = \int \sum_{m=1}^{M} |u_m|^2 + \left| \frac{\sum_{m=1}^{M} |u_m|\nabla|u_m|}{\left(\sum_{m=1}^{M} |u_m|^2\right)^{\frac{1}{2}}} \right|^2 \\ &\leqslant \int \sum_{m=1}^{M} |u_m|^2 + |\nabla|u_m||^2 = I(\hat{\mathbf{u}}). \end{aligned}$$

Let $a \leqslant 1$ be such that $J(au\mathbf{y}) = J(\hat{\mathbf{u}})$. Then

$$I(au\mathbf{y}) \leqslant I(u\mathbf{y}) \leqslant I(\hat{\mathbf{u}})$$

By the minimality of $\mathbf{u}$, the above inequalities must be equalities:

$$a = 1, \quad I(u\mathbf{y}) = I(\mathbf{u}).$$

Therefore $u\mathbf{y}$ is also a ground-state. Notice that $J(\mathbf{u}) = J(u\mathbf{y})$ implies that $\hat{\mathbf{u}}(x) = u(x)\mathbf{z}(x)$ a.e. $x \in \mathbb{R}^d$, where $\mathbf{z}(x) \in \mathcal{X}$.
Since $u\mathbf{y}$ is a bound-state for (M-NLS), one easily checks that

$$-\Delta u + u = f_{max}|u|^{2p}u$$

and so, setting $c = (f_{max})^{1/2p}$, cu is a bound-state for (NLS). The fact that $u\mathbf{y}$ is a ground-state clearly implies that $cu = Q(\cdot + b)$, for some $b \in \mathbb{R}^d$. From the maximum principle (see, for example, [32, Theorem 3.5]), $u > 0$ in $\mathbb{R}^d$.
Since $\hat{\mathbf{u}}(x) = u(x)\mathbf{y}(x)$ is a bound-state, inserting this expression into system (M-NLS), one obtains

$$2\nabla u \cdot \nabla y_m + u\Delta y_m = 0, \quad m = 1, ..., M$$

By integration by parts,

$$\begin{aligned}\int u y_m \nabla u \cdot \nabla y_m &= -\int u y_m \nabla u \cdot \nabla y_m - \int |u|^2 |\nabla y_m|^2 - \int |u|^2 y_m \Delta y_m \\ &= -\int u y_m \nabla u \cdot \nabla y_m - \int |u|^2 |\nabla y_m|^2 + 2\int u y_m \nabla u \cdot \nabla y_m \\ &= \int u y_m \nabla u \cdot \nabla y_m - \int |u|^2 |\nabla y_m|^2\end{aligned}$$

Hence

$$\int |u|^2 |\nabla y_m|^2 = 0, \quad m = 1, ..., M.$$

which, together with the smoothness of $\hat{\mathbf{u}}$ and u (cf. Proposition 1.2.5), implies that y_m is constant. Therefore

$$\hat{\mathbf{u}} = u\mathbf{y}, \mathbf{y} \in \mathcal{X}.$$

Finally, since $u > 0$, one may write $u_m(x) = |u_m(x)|e^{i\theta(x)} = u(x)\mathbf{y}e^{i\theta(x)}$. Then, since $I(\mathbf{u}) = I(\hat{\mathbf{u}})$,

$$\begin{aligned}\int \sum_{m=1}^{M} |u_m|^2 + |\nabla |u_m||^2 &= \int \sum_{m=1}^{M} |\hat{u}_m|^2 + |\nabla \hat{u}_m|^2 = I(\hat{\mathbf{u}}) = I(\mathbf{u}) = \int \sum_{m=1}^{M} |u_m|^2 + |\nabla u_m|^2 \\ &= \int \sum_{m=1}^{M} |u_m|^2 + |\nabla |u_m||^2 + |u_m|^2 |\nabla \theta_m(x)|^2.\end{aligned}$$

One then concludes that θ_m is constant, which ends the proof. □

Remark 2.2.3. The previous caracterization result is also valid when the domain is a bounded set Ω: one must simply replace $Q(\cdot + y)$ by a generic ground-state defined on Ω. Moreover, the fact that the constants appearing in (2.2.4) do not depend on Ω is a remarkable property. As a consequence, the question of whether G^* is empty or not is also independent of Ω. For example, by Theorem 2.2.3, we know that, for $M = 3$, $k_{12} > 0$, $k_{13}, k_{23} < 0$ and $\Omega = \mathbb{R}^d$, either $a_1 = a_2 = 0$ or $a_3 = 0$. This has been proven by arguing that translating the third component of a ground-state to the infinite decreases the action, which is not an available argument for Ω bounded. Now, however, we see that the result is also true for any Ω for which the existence of ground-states is known, in particular over bounded domains.

2.3 Examples

In this section, we apply the results to some special cases, obtaining in particular some already known results (in a more simplified way).

We start with $M = 2$. Given $(u, v) \in G^*$, we denote by a, b the constants of the characterization from Theorem 2.2.2.

Corollary 2.3.1 ([54])**.** *Suppose that* $k_{11} = k_{22} \leqslant 0$ *and* $k_{12} > -k_{11}$*. Let* $(u, v) \in G^*$*. Then* $a = b = (k_{11} + k_{12})^{-\frac{1}{2p}}$*.*

Proof. By Theorem 2.2.2, we know that

$$\begin{cases} k_{11}a^{2p} + k_{12}a^{p-1}b^{p+1} = 1 \\ k_{22}b^{2p} + k_{12}b^{p-1}a^{p+1} = 1 \end{cases}.$$

Suppose that $a \neq b$. By the symmetry of the system, it's enough to prove that $a \geqslant b$.

Multiplying the first equation by a^2, the second by b^2 and subtracting,

$$k_{11}a^{2p+2} - k_{22}b^{2p+2} = a^2 - b^2.$$

If $a < b$, the left-hand side is nonnegative and the right one is negative, which is absurd. Therefore $a = b$. The value of a can now be directly calculated from the system. □

Corollary 2.3.2 ([74])**.** *Suppose that* $p = 1$ *and* $k_{jm} > 0$*,* $j, m = 1, 2$*. Then*

1. *If* $k_{11} \neq k_{22}$ *and* $k_{11} \leqslant k_{12} \leqslant k_{22}$*,* $G^* = \varnothing$*;*

2. *If* $k_{12} \notin [\min\{k_{11}, k_{22}\}, \max\{k_{11}, k_{22}\}]$ *and* $(u, v) \in G^*$*, then*

$$a = \sqrt{\frac{k_{22} - k_{12}}{k_{11}k_{22} - k_{12}^2}}, \quad b = \sqrt{\frac{k_{11} - k_{12}}{k_{11}k_{22} - k_{12}^2}}. \tag{2.3.1}$$

   Consequently, $G^* = \varnothing$ *if* $k_{12} < \min\{k_{11}, k_{22}\}$ *and* $G^* = G$ *if* $k_{12} > \max\{k_{11}, k_{22}\}$*.*

3. *If* $k_{11} = k_{12} = k_{22}$*,* $(u, v) \in G^*$ *if and only if*

$$(a, b) = \left(\frac{1}{\sqrt{k_{11}}}\cos\alpha, \frac{1}{\sqrt{k_{11}}}\sin\alpha\right), \ \alpha \in]0, \pi/2[.$$

Proof. By Theorem 2.2.2, we know that

$$\begin{cases} k_{11}a^2 + k_{12}b^2 = 1 \\ k_{22}b^2 + k_{12}a^2 = 1 \end{cases}$$

Therefore

$$\begin{cases} (k_{11}k_{22} - k_{12}^2)a^2 = k_{22} - k_{12} \\ (k_{11}k_{22} - k_{12}^2)b^2 = k_{11} - k_{12} \end{cases}$$

1. If $k_{11} \neq k_{22}$ and $k_{11} \leqslant k_{12} \leqslant k_{22}$, suppose, without loss of generality, that $k_{11} < k_{12}$. Then
$$\frac{a^2}{b^2} = \frac{k_{22} - k_{12}}{k_{11} - k_{12}} \leqslant 0$$
which is absurd.

2. If $k_{12} \notin [\min\{k_{11}, k_{22}\}, \max\{k_{11}, k_{22}\}]$, one can explicitly determine the values of a and b, thus obtaining (2.3.1). Suppose, without loss of generality, that $k_{11} \leqslant k_{22}$. If $k_{12} < k_{22}$, then one easily checks that
$$I(aQ, bQ) = \frac{k_{22} - k_{12}}{k_{11}k_{22} - k_{12}^2} + \frac{k_{11} - k_{12}}{k_{11}k_{22} - k_{12}^2} > \frac{1}{k_{22}} = I\left(\frac{1}{\sqrt{k_{22}}}Q\right).$$
Therefore $G^* = \varnothing$. If $k_{12} > k_{22}$, the above inequality is reversed and one obtains $G = G^*$.

3. If $k_{11} = k_{12} = k_{22}$, then $a^2 + b^2 = 1/k_{11}$ and so there exists $\alpha \in]0, \pi/2[$ such that
$$(a, b) = \left(\frac{1}{\sqrt{k_{11}}}\cos\alpha, \frac{1}{\sqrt{k_{11}}}\sin\alpha\right).$$
On the other hand, any pair of this form is in S^+ and has minimal norm. The conclusion follows from Theorem 2.2.2.

□

Corollary 2.3.3. *Suppose that* $k_{11} = k_{22} > 0$, $k_{12} > 0$. *and* $(p-1)(pk_{11} - k_{12}) > 0$. *If* $(u, v) \in G^*$, *then* $a_0 = b_0 = (k_{11} + k_{12})^{-\frac{1}{2p}}$.

Proof. Again by Theorem 2.2.2,
$$\begin{cases} k_{11}a^{2p} + k_{12}a^{p-1}b^{p+1} = 1 \\ k_{22}b^{2p} + k_{12}b^{p-1}a^{p+1} = 1 \end{cases}$$
Taking the difference between the two equations and dividing by b^{2p},
$$k_{11}\left(\frac{a}{b}\right)^{2p} - k_{11} + k_{12}\left(\left(\frac{a}{b}\right)^{p-1} - \left(\frac{a}{b}\right)^{p+1}\right) = 0.$$
Consider the function $f(x) = k_{11}x^{2p} - k_{11} + k_{12}(x^{p-1} - x^{p+1}),\ x > 0$. It is clear that $f(1) = 0$ and $f(0) < 0$. We want to see that f does not have zeroes on both sides of 1. One has
$$\begin{aligned} f'(x) &= 2pk_{11}x^{2p-1} + k_{12}((p-1)x^{p-2} - (p+1)x^p) \\ &= x^{p-2}\left(2pk_{11}x^{p+1} + k_{12}((p-1) - (p+1)x^2)\right) =: x^{p-2}g(x) \end{aligned}$$
and
$$g'(x) = 2p(p+1)k_{11}x^p - 2(p+1)k_{12}x.$$

Clearly

$$g'(x) = 0 \Leftrightarrow x = \left(\frac{k_{12}}{pk_{11}}\right)^{\frac{1}{p-1}}.$$

Since g' has a unique zero, f has at most three (counting multiplicities), one of which $x = 1$. If $p > 1$, since $f(x) \to \infty$ when $x \to \infty$ and $f'(1) = g(1) = 2(pk_{11} - k_{12}) > 0$, all the zeroes of f have to be on the same side with respect to $x = 1$, as we wanted. If $p < 1$, since $f(x) \to -\infty$ when $x \to \infty$ and $f'(1) = g(1) = 2(pk_{11} - k_{12}) < 0$, we obtain the same conclusion.

Suppose, without loss of generality, that f has no zeroes on $]0,1[$. It follows that $f(x) = 0$ implies $x \leqslant 1$ and so $a \leqslant b$. By the symmetry of the system, $a \geqslant b$. Hence $a = b$. The value of a can now be determined from the system. □

Remark 2.3.1. In the case $p < 1$ and $pk_{11} - k_{12} > 0$, one may easily check that the function f in the above proof has three distinct zeroes $x_0, 1$ and x_0^{-1}.

Proposition 2.3.4. *Fix $M \geqslant 2$, $p \geqslant 1$ and suppose that, for each $1 \leqslant m \leqslant M$, $k_{mm} > 0$. If $\beta = \max_{m \neq j} |k_{jm}|$ is sufficiently small, then, letting $\mathcal{I}$ be the set of m_0's such that $k_{m_0m_0} = \max_m k_{mm}$ and, for any $m_0 \in \mathcal{I}$, $\mathcal{Q}_{m_0} \in (H^1(\mathbb{R}^d))^M$ defined by $(\mathcal{Q})_m = 0$ if $m \neq m_0$ and $(\mathcal{Q})_{m_0} = k_{m_0m_0}^{-\frac{1}{p+1}} Q$ (recall Lemma 1.2.2), one has*

$$G = \{e^{i\theta}\mathcal{Q}_{m_0}(\cdot + y),\ m_0 \in \mathcal{I}, \theta \in \mathbb{R}, y \in \mathbb{R}^d\}. \tag{2.3.2}$$

Proof. Set $\mathbf{a}_0 = (k_{mm}^{-\frac{1}{2p+2}})_{1 \leqslant m \leqslant M}$ and $\mathcal{S}_0$ the vector space of symmetrical matrices $M \times M$ with zero diagonal, equipped with the l^∞ norm. Consider $F : \mathcal{S}_0 \times \mathbb{R}^M \to \mathbb{R}^M$,

$$F_m(D, \mathbf{a}) = k_{mm} a_m^{2p+2} + \sum_{j=1,\ m \neq j}^{M} d_{jm} a_m^{p-1} a_j^{p+1} - 1,\ D = (d_{jm}),\ \mathbf{a} = (a_m)_{1 \leqslant m \leqslant M}.$$

Then $F(0, \mathbf{a}_0) = 0$, F is C^1 and it is easy to see that the jacobian of F with respect to $\mathbf{a}$ in $\mathbf{a}_0$ is nonzero. By the Implicit Function Theorem, if $\|D\|_{\mathcal{S}_0} < \delta$, there exists a unique solution of $F(D, \mathbf{a}) = 1$, called $\mathbf{a}(D)$, and there exists $\epsilon > 0$ small enough such that $\|\mathbf{a}(D) - \mathbf{a}_0\|_{\mathbb{R}^M} < \epsilon$. Consequently

$$\sum_{m=1}^{M} (\mathbf{a}(D))_m^2 \geqslant \sum_{m=1}^{M} (\mathbf{a}_0)_m^2 - \epsilon^2 = \sum_{m=1}^{M} k_{mm}^{-\frac{1}{p+1}} - \epsilon^2 > \min_m \{k_{mm}^{-\frac{1}{p+1}}\},$$

for ϵ small. Moreover, since $p \geqslant 1$, one easily checks that, for a fixed $\eta > 0$ large, when β is sufficiently small, any solution of $F(D, \mathbf{a}) = 1$ with $\|\mathbf{a}\|_{\mathbb{R}^M} < \eta$ must satisfy $\|\mathbf{a} - \mathbf{a}_0\|_{\mathbb{R}^M} < \epsilon$.

If there existed $\mathbf{u} \in G^*$, by Theorem 2.2.4, $\mathbf{u}$ would be of the form

$$\mathbf{u} = (a_m e^{i\theta_m} Q(\cdot + y))_{1 \leqslant m \leqslant M},\ \mathbf{a} = (a_m)_{1 \leqslant m \leqslant M} \in S^+$$

and, arguing as in Theorem 2.2.2, $\mathbf{a}$ would be a solution of (2.2.2). From Theorem 2.2.4, one may easily see that, for η large, $\|\mathbf{a}\|_{\mathbb{R}^M} < \eta$. By uniqueness, $\mathbf{a} = \mathbf{a}(D)$, for β small. Therefore

$$I(Q) < \sum_{m=1}^{M} (\mathbf{a}(D))_m^2 I_1(Q) = I(\mathbf{u})$$

which contradicts $\mathbf{u} \in G$. Therefore G^* is empty.

If there exists $\mathbf{u} \in G$ with at least two nonzero components, the vector of nonzero components of $\mathbf{u}$, $\mathbf{u}^+$, has to be a fully nontrivial ground-state for a (L-NLS) system, with $2 \leqslant L \leqslant M$. Applying the above argument, we obtain a contradiction. Therefore any ground-state has exactly one nontrivial component, which must be a scalar multiple of Q. A simple comparison of the action of such solutions proves the characterization (2.3.2). $\square$

Remark 2.3.2. In [47], the case $\Omega = \mathbb{R}^d$, $M = 3$, $p = 1$, $k_{12}, k_{23} > 0$ and $k_{13} < 0$ is considered. The authors prove that if $k_{mm} = 1$, $m = 1, 2, 3$, $k_{12}, k_{23} \approx \delta^2$ and $k_{13} \approx -\delta$, $\delta > 0$ small, any fully nontrivial ground-state is not radial. This conclusion is actually a corollary from the previous proposition: in fact, it implies that $G^* = \varnothing$.

Example 2.3.1. Consider $M = 3$, $p = 1$ and suppose that the coefficient matrix K is of the form

$$K = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}, \quad a \leqslant b \leqslant c \in \mathbb{R}\backslash\{0\}.$$

Now one must divide in several cases:

- $c < 0$: in this case, the condition for the existence of ground-states is not verified and so there are no ground-states;
- $c > 0, b < 0$: applying Theorem 2.2.3, any ground-state $\mathbf{u}$ satisfies either $u_1 = 0$ or $u_2, u_3 = 0$. The second possibility implies that u_1 satisfies $-\Delta u_1 + u_1 = 0$, which is impossible. Therefore $u_1 = 0$. Since $\mathbf{u} = (a_m Q(\cdot + y))_{1 \leqslant m \leqslant 3}$, a direct substitution on the system gives $a_2 = a_3 = c^{-1/2}$.
- $b > 0$: suppose that $\mathbf{u}$ is a fully nontrivial ground-state. Then, inserting the characterization formula (2.2.4) in the system (M-NLS), we obtain

$$K\mathbf{y} = (1, 1, 1)^T, \quad \mathbf{y} = (a_1^2, a_2^2, a_3^2)^T. \tag{2.3.3}$$

  Thus

$$a_1^2 = \frac{a+b-c}{2ab},\ a_2^2 = \frac{a+c-b}{2ac},\ a_3^2 = \frac{b+c-a}{2bc}.$$

  This implies that $(a+b-c)a > 0$ and $(a+c-b)a > 0$. Now, if $\mathbf{v}$ is a semitrivial ground-state, using the characterization and the fact that $c \geqslant a, b$,

$$\mathbf{v} = \left(0, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}}\right) u_0.$$

Now, comparing the actions of these two solutions, the condition for the existence of fully nontrivial ground-states is

$$\frac{-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac}{2ab} \leqslant 2$$

which, for $a > 0$, simplifies to

$$2c(a+b) \leqslant (a+b)^2 + c^2, i.e., (a+b-c)^2 \geqslant 0. \tag{2.3.4}$$

Therefore, for $a > 0$, one has the following:

- if $a + b \leqslant c$, $G^* = \varnothing$, since system (2.3.3) has no positive solutions. This implies
$$G = \left\{ \left(0, \frac{e^{i\theta_1}}{\sqrt{c}}, \frac{e^{i\theta_2}}{\sqrt{c}}\right) Q(\cdot + y) : \theta_1, \theta_2 \in \mathbb{R},\ y \in \mathbb{R}^d \right\}.$$
- if $a + b > c$, then
$$G = \left\{ \left(e^{i\theta_1}\sqrt{\frac{a+b-c}{2ab}}, e^{i\theta_2}\sqrt{\frac{a+c-b}{2ac}}, e^{i\theta_3}\sqrt{\frac{b+c-a}{2bc}} \right) Q(\cdot + y) : \theta_1, \theta_2, \theta_3 \in \mathbb{R},\ y \in \mathbb{R}^d \right\}.$$

For $a < 0$, inequality (2.3.4) is reversed and strict, hence $G^* = \varnothing$.

Hence the necessary and sufficient condition for the existence of fully nontrivial ground-states with $a \leqslant b \leqslant c \in \mathbb{R}$ is $a + b > c$.

EXAMPLE 2.3.2. Consider $M = 3$, $p = 1$ and suppose that the coefficient matrix K is of the form

$$K = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}, \quad 1 \ll a \leqslant b \leqslant c$$

The previous example may be seen as a limit when a, b, c are very large. With some computations, one derives the following:

- The possible semitrivial ground-state is given by
$$\mathbf{v} = \left(0, \frac{1}{\sqrt{1+c}}, \frac{1}{\sqrt{1+c}}\right).$$
- The possible fully nontrivial ground-state, $\mathbf{u} = (a_m u_0)_{1 \leqslant m \leqslant 3}$, is given by
$$a_1^2 = \frac{1 + (a+b)c - a - b - c^2}{1 + 2abc - a^2 - b^2 - c^2},\ a_2^2 = \frac{1 + (a+c)b - a - c - b^2}{1 + 2abc - a^2 - b^2 - c^2},$$

$$a_3^2 = \frac{1 + (c+b)a - c - b - a^2}{1 + 2abc - a^2 - b^2 - c^2}.$$

We assume that a, b, c are such that all numerators and denominators above are positive. Notice that this is true for a, b, c large enough and $a + b > c + 1$.

As in the previous example, if one compares the corresponding action levels, one has $G^* \neq \varnothing$ if and only if

$$0 \leqslant (a + b - c)(a + b - c - 2).$$

Since we assumed that $a + b > c + 1$, the condition is simply $a + b \geqslant c + 2$. We see that, even for systems where the couplings $k_{jm}, m \neq j$, are large comparing to the diagonal terms k_{mm}, one may have $G^* = \varnothing$. This example shows that, in order for one to have $G^* \neq \varnothing$, one must take into account the relation between coupling coefficients. This does not go against the conclusion of Corollary 3.1.8 and the perturbation arguments of Chapter 2: the problem here is that a, b and c are not close to each other.

EXAMPLE 2.3.3. Consider system (3-NLS), $p = 1$ and the coefficient matrix

$$K = \begin{bmatrix} 0 & b & 1 \\ b & 0 & 2 \\ 1 & 2 & \mu \end{bmatrix}, \quad b > 0, \mu \in \mathbb{R}.$$

Using the characterization result, everything is reduced to the study of the proportionality constants a_1, a_2 and a_3. For the sake of simplicity, $x = a_1^2$, $y = a_2^2$, $z = a_3^2$. It is now a simple calculation to obtain the following:

- Semitrivial A: The possible ground-state with $x = 0$ satisfies $y = (2 - \mu)/4$, $z = \frac{1}{2}$. This solution only exists if $2 > \mu$.
- Semitrivial B: Analogously, the possible ground-state with $x = 0$ satisfies $x = 1 - \mu$, $z = 1$. This solution only exists if $1 > \mu$.
- Semitrivial C: The possible ground-state with $z = 0$ satisfies $x = y = 1/b$;
- Semitrivial D: If $x = y = 0$, then $z = 1/\mu$. This solution only exists if $\mu > 0$;
- Nontrivial E: For the possible fully nontrivial ground-state,

$$x = \frac{\mu b - 2b + 2}{b(\mu b - 4)}, y = \frac{\mu b - b - 1}{b(\mu b - 4)}, \quad z = \frac{b - 3}{\mu b - 4}.$$

  This solution exists only if $b > 3$ and $\mu > 2(b-1)/b$ or if $b < 3$ and $\mu < 2(b-1)/b$.

The action for each of these solutions is (up to a constant)

$$A : \frac{4 - \mu}{4}, \quad B : 2 - \mu, \quad C : 2/b, \quad D : 1/\mu, \quad E : \frac{b^2 + (2\mu - 6)b + 1}{b(\mu b - 4)}$$

Now we compare the various actions, whenever the solutions exist:

1. First of all, A is always lower than B (when B exists). Therefore we may discard this solution;

2. The solution D is the best one if $\mu > 2, b/2$;

3. A tideous computation shows that E is the ground-state if $b < 3$ and $\mu < 2(b-1)/b$;

4. In the remaining area, A is better than C if $2 > \mu > 4 - 8/b$.

Intersecting these comparisons with the domains where each solution exists, we obtain diagram 2, which is already revealing of the complexity of this problem.

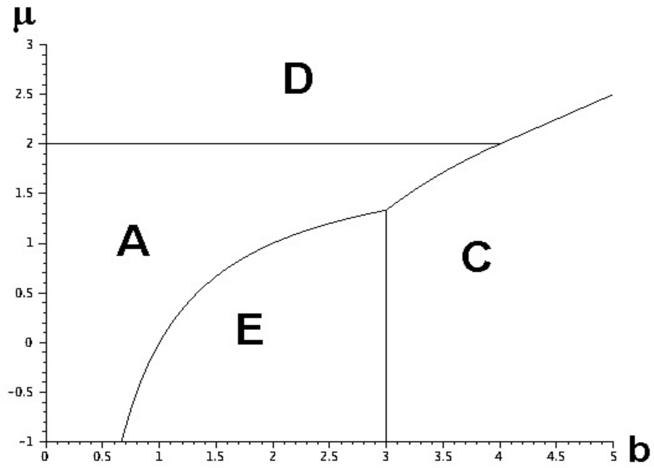

Figure 2.1: Regions of the $b - \mu$ plane where each solution is a ground-state.

Several remarks are necessary:

1. First of all, we see that, for $b < 3$, when $-\mu$ is very large, the ground-state is fully nontrivial. Moreover, if $b > 3$, no value of μ produces fully nontrivial ground-states;

2. One might think that some solutions (for example, the fully nontrivial one), if they exist, would always have minimal action. However, the reader may check that this is not true for this system;

3. It is natural that one never has $u_1 \neq 0$ and $u_2 = 0$, since the coupling coefficients associated with the second component are larger than the coefficients associated with the first component. In fact, this is a consequence of the monotonicity with respect to the coefficient matrix (see Lemma 3.1.3).

Chapter 3

Incoherent ground-states

Throughout this chapter, we consider ground-states of the system

$$\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} k_{jm} |u_j|^{p+1} |u_m|^{p-1} u_m = 0, \quad \omega_m > 0, \ m = 1, ..., M, \qquad \text{(M-NLS')}$$

and focus on the question of whether ground-states are fully nontrivial or semitrivial[1].

3.1 Perturbation theory

We start out by deriving monotonicity properties for the ground-state action level with respect to the parameters of our system.

Definition 3.1.1. *Given $A, B \in \mathbb{R}^{k \times l}$, we say that $A \leqslant B$ if*

$$a_{jm} \leqslant b_{jm}, \quad 1 \leqslant m \leqslant k, 1 \leqslant j \leqslant l, \quad A = (a_{jm}), B = (b_{jm}).$$

Lemma 3.1.2 (Monotonicity of the action with respect to ω)**.** *Let $\omega = (\omega_1, ..., \omega_M)$ and $\omega' = (\omega'_1, ..., \omega'_M)$ be such that $\omega \geqslant \omega'$. Fix a matrix $K = (k_{jm})_{1 \leqslant m, j \leqslant M} \in \mathbb{R}^{M \times M}$. Let $\mathbf{u}^{\omega}$ be a ground-state of*

$$\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} k_{jm} |u_j|^{p+1} |u_m|^{p-1} u_m = 0 \quad m = 1, ..., M$$

and $\mathbf{u}^{\omega'}$ be a ground-state of

$$\Delta u_m - \omega'_m u_m + \sum_{j=1}^{M} k_{jm} |u_j|^{p+1} |u_m|^{p-1} u_m = 0 \quad m = 1, ..., M.$$

Then $J(\mathbf{u}^{\omega}) \geqslant J(\mathbf{u}^{\omega'})$.

[1] We recall that the existence of incoherent ground-states follows the same lines as the existence of ground-states for (M-NLS).

Proof. Simply recall that

$$
\begin{aligned}
J(\mathbf{u}^{\omega}) &= \left(\inf_{J(\mathbf{u})=1}\sum_{m=1}^{M}\omega_m\|\mathbf{u}_m\|_2^2+\|\nabla\mathbf{u}_m\|_2^2\right)^{\frac{p+1}{p}}\\
&\geqslant \left(\inf_{J(\mathbf{u})=1}\sum_{m=1}^{M}\omega'_m\|\mathbf{u}_m\|_2^2+\|\nabla\mathbf{u}_m\|_2^2\right)^{\frac{p+1}{p}} = J(\mathbf{u}^{\omega'}).
\end{aligned}
$$

□

Analogously, we may obtain the following:

Lemma 3.1.3 (Monotonicity of the action with respect to K)**.** *Fix $\omega \in (\mathbb{R}^+)^M$. Consider matrices $K = (k_{jm})_{1\leqslant m,j\leqslant M} \in \mathbb{R}^{M\times M}$ and $K' = (k'_{jm})_{1\leqslant m,j\leqslant M} \in \mathbb{R}^{M\times M}$ such that $K \geqslant K'$. Let $\mathbf{u}^K$ be a ground-state of*

$$
\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0 \quad m = 1, ..., M
$$

and $\mathbf{u}^{K'}$ be a ground-state of

$$
\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} k'_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0 \quad m = 1, ..., M.
$$

Then $I(\mathbf{u}^{K'}) \geqslant I(\mathbf{u}^K)$.

Suppose that one wishes to study G^* in function of a given set of couplings. Let P a nonempty symmetric subset of $\{1, ..., M\}^2$ (the set of ordered couples with entries in $\{1, ..., M\}$) and fix a matrix $K \in \mathbb{R}^{M^2}$. Given $\beta \in \mathbb{R}$, consider the system

$$
\Delta u_m - \omega_m u_m + \sum_{(j,m)\notin P} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m + \sum_{(j,m)\in P} \beta k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0, \quad (3.1.1)
$$

where $m = 1, \ldots, M$. Suppose, for the sake of simplicity, that $k_{jm} > 0, (j, m) \in P$. Everytime a functional, a set or a solution depends on β, we shall place a subscript β.

Let $\mathcal{I}_\beta$ be the ground-state action level for each β (recall (2.1.1)),

$$
\mathcal{I}_\beta = \left(\inf_{J_\beta(\mathbf{u})=1} I(\mathbf{u})\right)^{\frac{p+1}{p}}.
$$

For any $X \subset \{1, ..., M\}$, define

$$
\mathcal{I}_\beta^X := \left(\inf_{J_\beta(\mathbf{u})=1,\mathbf{u}_m=0,m\notin X} I(\mathbf{u})\right)^{\frac{p+1}{p}}, \quad \mathcal{I}_\beta^{sem} := \min\{\mathcal{I}_\beta^X : X \subsetneq \{1, ..., M\}\}.
$$

Notice that $\mathcal{I}_\beta = \mathcal{I}_\beta^{\{1,...,M\}}$. Then $G^* = G$ if and only if $\mathcal{I}_\beta < \mathcal{I}_\beta^{sem}$.

From the results regarding existence of ground-states, we know that, for each $X \subset \{1, ..., M\}$, there exists $\underline{\beta}_X \in \overline{\mathbb{R}}$ such that $\beta \leqslant \underline{\beta}_X$ if and only if $\mathcal{I}_\beta^X = +\infty$. Define

$$\underline{\beta}^{sem} := \min_{X \subsetneq \{1,...,M\}} \underline{\beta}_X, \quad \underline{\beta} := \underline{\beta}_{\{1,...,M\}}.$$

Then

1. If $\beta \leqslant \underline{\beta}$, there are no ground-states;
2. If $\underline{\beta} < \beta \leqslant \underline{\beta}^{sem}$, all ground-states are fully nontrivial;
3. If $\underline{\beta}^{sem} < \beta$, both $\mathcal{I}_\beta$ and $\mathcal{I}_\beta^{sem}$ are finite.

Proposition 3.1.4. *For any $X \subset \{1, ..., M\}$, the mapping $\beta \mapsto \mathcal{I}_\beta^X, \beta \in \mathbb{R}$, is continuous (in $\overline{\mathbb{R}}$). In particular, $\mathcal{I}_\beta$ and $\mathcal{I}_\beta^{sem}$ are continuous with respect to β.*

Proof. Notice that we only need to prove the proposition for $X = \{1, ..., M\}$, since any other case may be reduced to this one.
Fix $\beta_0 \in \mathbb{R}$. If $\beta_0 < \underline{\beta}$, then $\mathcal{I}_\beta \equiv +\infty$ in a neighbourhood of β_0 and so it is continuous. If $\beta_0 > \underline{\beta}$, let $\beta_n \to \beta_0$. By definition, there exists $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset (H^1(\mathbb{R}^d))^M$ such that

$$I(\mathbf{u}_n) = \mathcal{I}_{\beta_n}, \quad J_{\beta_n}(\mathbf{u}_n) = 1.$$

Let $\lambda_n = J_{\beta_0}(\mathbf{u}_n)^{-1/2p}$. Then $J_{\beta_0}(\lambda_n \mathbf{u}_n) = 1$. Moreover,

$$|\lambda_n^{-1/2p} - 1| = |J_{\beta_0}(\mathbf{u}_n) - J_{\beta_n}(\mathbf{u}_n)| = |\beta_n - \beta_0||J_P(\mathbf{u}_n)|$$

Since

$$|J_P(\mathbf{u}_n)| \leqslant C\|\mathbf{u}_n\|_{H^1}^{2p+2} \leqslant CI(\mathbf{u}_n)^{2p+2} \leqslant C(\mathcal{I}_{\beta_n})^{2p+2} < C,$$

we obtain $\lambda_n \to 1$. Therefore,

$$\liminf \mathcal{I}_{\beta_n} = \liminf I(\mathbf{u}_{\beta_n}) = \liminf I(\lambda_n \mathbf{u}_{\beta_n}) \geqslant \mathcal{I}_{\beta_0}.$$

On the other hand, for $n > 0$, let $\mathbf{u}$ be such that

$$\mathcal{I}_{\beta_0} = I(\mathbf{u}), \quad J_{\beta_0}(\mathbf{u}) = 1.$$

Define $\lambda^n = J_{\beta_n}(\mathbf{u})^{-1/2p}$. As before, $J_{\beta_n}(\lambda^n \mathbf{u}) = 1$ and $\lambda^n \to 1$. Hence

$$\mathcal{I}_{\beta_0} = I(\mathbf{u}) = \lim I(\lambda^n \mathbf{u}) \geqslant \limsup \mathcal{I}_{\beta_n}.$$

Therefore $\mathcal{I}_\beta$ is continuous for $\beta > \underline{\beta}$.
If $\beta_0 = \underline{\beta}$ and $\beta_n \to \beta_0^+$, consider $\mathbf{u}_n$ as above. Then, since $J_{\beta_0} \leqslant 0$,

$$1 = J_{\beta_n}(\mathbf{u}) = J_{\beta_0}(\mathbf{u}_n) + (\beta_n - \beta_0)J_P(\mathbf{u}_n) \leqslant C(\beta_n - \beta_0)I(\mathbf{u}_n)^{2p+2}.$$

and so $\mathcal{I}_{\beta_n} = I(\mathbf{u}_n) \to \infty = \mathcal{I}_{\beta_0}$. □

Lemma 3.1.5. *Suppose that $\underline{\beta} < \underline{\beta}^{sem}$. For β sufficiently close to $\underline{\beta}^{sem}$, $G = G^*$.*

Proof. Since $\underline{\beta} < \underline{\beta}^{sem} =: \beta_0$, we have

$$\mathcal{I}_{\beta_0} < \infty, \quad \mathcal{I}^{sem}_{\beta_0} = \infty,$$

which implies that $\mathcal{I}_{\beta_0} < \mathcal{I}^{sem}_{\beta_0}$. By continuity, the same inequality must be true for β close to β_0 and so $G = G^*$.

□

Corollary 3.1.6. *Consider system* (M-NLS').

1. *If $M = 2$ and $0 < k_{11}, k_{22} \ll k_{12}$, $G = G^*$;*
2. *For $M \geqslant 3$, if $k_{mm} = -1$, $\forall m$ and $k_{jm} = \beta$, $\forall m \neq j$, there exists $\epsilon > 0$ such that, if*

$$\frac{2}{M-1} < \beta < \frac{2}{M-2} + \epsilon,$$

then $G = G^*$.

Proof. In the first case, take $P = \{(1,1),(2,2)\}$. One easily observes that $\underline{\beta}^{sem} = 0$ and that $\underline{\beta} < 0$. Therefore, using the previous lemma, for $\beta > 0$ small enough, $G = G^*$.

To prove the second case, take $P = \{(j,m),\ 1 \leqslant m, j \leqslant M,\ m \neq j\}$. A simple calculation shows that $\underline{\beta}(M) = 2/(M-1)$ and $\underline{\beta}^{sem}(M) = \underline{\beta}(M-1) = 2/(M-2)$. Therefore, by the previous lemma, there exists $\epsilon > 0$ such that, for $2/(M-1) < \beta < 2/(M-2) + \epsilon$, $G = G^*$.

□

Now, we argue briefly that the same procedure may be applied to study the dependence of G^* on $\omega = (\omega_1, ..., \omega_M)$. Suppose that $\omega_m > 1, \forall m$ (this condition is not restraining at all, since any case may be reduced to this one by a simple scaling). Define

$$\underline{\eta} = -\min_{1\leqslant m\leqslant M} 1/(\omega_m - 1).$$

For $\eta > \underline{\eta}$, consider the system

$$\Delta u_m - (1 + \eta(\omega_m - 1))u_m + \sum_{j=1}^{M} k_{jm}|u_j|^{p+1}|u_m|^{p-1}u_m = 0, \quad m = 1, ..., M.$$

Now we write the dependence on η as a superscript. If one defines

$$\mathcal{I}^\eta = \left(\inf_{J(\mathbf{u})=1} I^\eta(\mathbf{u})\right)^{\frac{p+1}{p}},$$

and, for any $X \subset \{1, ..., M\}$,

$$(\mathcal{I}^\eta)^X := \left(\inf_{J(\mathbf{u})=1, \mathbf{u}_m=0, i\notin X} I^\eta(\mathbf{u})\right)^{\frac{p+1}{p}}, \quad (\mathcal{I}^\eta)^{sem} := \min_{X \subsetneq \{1,...,M\}} (\mathcal{I}^\eta)^X,$$

we have once again $G = G^*$ if and only if $\mathcal{I}^\eta < (\mathcal{I}^\eta)^{sem}$. As before, we may show that

Proposition 3.1.7. *For any $X \subset \{1, ..., M\}$, the mapping $\eta \mapsto (\mathcal{I}^\eta)^X, \eta > \underline{\eta}$, is continuous. In particular, $\mathcal{I}^\eta$ and $(\mathcal{I}^\eta)^{sem}$ are continuous with respect to η.*

Corollary 3.1.8. *Consider system (M-NLS'), $M \geqslant 3$. Suppose that $p \leqslant 1$, $\omega_1 \leqslant \omega_2 \leqslant ... \leqslant \omega_M$ and $k_{jm} = b > 0$, $\forall m \neq j$. Assume that*

$$\frac{M-1}{(M-2)^{1/p}} > \frac{M}{(M-1)^{1/p}} \left(\frac{\omega_M}{\omega_1}\right)^{\frac{2-p(N-2)}{2p}}. \tag{3.1.2}$$

Then there exists $\delta > 0$ such that, if $\max_m |k_{mm}| < \delta b$, $G = G^*$.

Proof. First of all, notice that, if $\mathbf{u}$ is a ground-state, $\mathbf{v} = b^{1/2p}\mathbf{u}$ is a ground-state of

$$\Delta u_m - \omega_m u_m + \sum_{j=1}^{M} \frac{k_{jm}}{b} |u_j|^{p+1} |u_m|^{p-1} u_m = 0, \quad m = 1, ..., M.$$

Therefore, we may consider that $b = 1$ and that the diagonal terms are small. Then such a system may be seen as a β-perturbation of

$$\Delta u_m - \omega_m u_m + \sum_{j=1, m\neq j}^{M} |u_j|^{p+1} |u_m|^{p-1} u_m = 0, \quad m = 1, ..., M.$$

We denote by $\mathcal{I}(\omega_1, ..., \omega_M)$ the corresponding ground-state action level. By the monotonicity properties, $\mathcal{I}(\omega_1, ..., \omega_M) \leqslant \mathcal{I}(\omega_M, ..., \omega_M)$. On the other hand, if $\mathcal{I}^{sem}(\omega_1, ..., \omega_M)$ is the semitrivial ground-state action level (that is, the lowest action among semitrivial bound-states), then $\mathcal{I}^{sem}(\omega_1, ..., \omega_M) \geqslant \mathcal{I}^{sem}(\omega_1, ..., \omega_1)$. The proof will be concluded if one proves that

$$\mathcal{I}^{sem}(\omega_1, ..., \omega_1) > \mathcal{I}(\omega_M, ..., \omega_M).$$

Using a suitable scaling, we have

$$\mathcal{I}^{sem}(\omega_1, ..., \omega_1) = \omega_1^{\frac{2-p(N-2)}{2p}} \mathcal{I}^{sem}(1, ..., 1), \tag{3.1.3}$$
$$\mathcal{I}(\omega_M, ..., \omega_M) = \omega_M^{\frac{2-p(N-2)}{2p}} \mathcal{I}(1, ..., 1).$$

Therefore we only have to compare the ground-state and semitrivial ground-state actions levels for

$$\Delta u_m - u_m + \sum_{j=1, m\neq j}^{M} |u_j|^{p+1} |u_m|^{p-1} u_m = 0, \quad m = 1, ..., M. \tag{3.1.4}$$

We claim that any fully nontrivial ground-state of (3.1.4) must be of the form $\mathbf{u} = (u_m)_{1\leqslant m\leqslant M}$, with $u_m = (M-1)^{-\frac{1}{2p}} u_0$, where u_0 is a scalar ground-state: by Theorem 2.2.4,

$$u_m = a_m e^{i\theta_m} u_0, \quad a_m > 0, \theta_m \in \mathbb{R}$$

Inserting this information in the system, we have

$$a_m^{p-1}\sum_{j\neq m} a_j^{p+1} = 1,\ m = 1, ..., M.$$

Suppose, without loss of generality, that $a_1 < a_2$ and $a_1 \leqslant a_m, i \geqslant 2$. Then

$$\begin{cases} \sum_{j\neq 1} a_j^{p+1} = a_1^{1-p} \\ \sum_{j\neq 2} a_j^{p+1} = a_2^{1-p} \end{cases}$$

This is a contradiction, since the first sum is larger than the second. Therefore $a_1 = ... = a_M = a$ and so

$$a^{2p}(M-1) = 1,$$

yielding the claim. Now compute the action for such a ground-state:

$$I(\mathbf{u}) = \frac{M}{(M-1)^{\frac{1}{p}}} I(u_0). \tag{3.1.5}$$

Since the mapping $M \mapsto M/(M-1)^{\frac{1}{p}}$ is strictly decreasing and any semitrivial ground-state is a fully nontrivial ground-state for the same system with $M - L$ equations, for some $L \in \mathbb{N}$, we have

$$\mathcal{I}(1, ..., 1) = \frac{M}{(M-1)^{\frac{1}{p}}} I(u_0)$$

$$\mathcal{I}^{sem}(1, ..., 1) = \min_{1\leqslant L\leqslant M-2} \left\{ \frac{M-L}{(M-L-1)^{\frac{1}{p}}} I(u_0) \right\} = \frac{M-1}{(M-2)^{\frac{1}{p}}} I(u_0).$$

The result follows from (3.1.3) and hypothesis (3.1.2). □

Notice that the first part of Corollary 3.1.6 is already known (see [56] and [55]). Also, in Corollary 3.1.8, if $\omega_1 = \omega_M$, the result is a particular case of [52] and [62]. Even so, we prove these results for two reasons: first, the proof is very simple when one looks from this pertubative perspective; second, the approach is rather different in nature and it deals only with continuity properties, which may have a greater capacity of generalization to other systems.

Regarding Corollary 3.1.8, a comment is in need: it might be expected that the restriction $p \leqslant 1$ would be technical. In fact, for $M = 2$, the result is valid for any $p > 0$. By contrast, we prove

Proposition 3.1.9. *Consider system (3-NLS), with $p = 3$, $k_{mm} = 0, \forall m$, and $k_{jm} = 1, \forall m \neq j$. Then $G^* = \varnothing$.*

Proof. First of all, using the characterization result, any fully nontrivial ground-state is of the form $\mathbf{u} = (a_m u_0)_{1\leqslant m\leqslant 3}$, with u_0 a scalar ground-state. Inserting this formula in the system and writing $b_m = a_m^2$,

$$\begin{cases} b_1(b_2^2 + b_3^2) = 1 \\ b_2(b_1^2 + b_3^2) = 1 \\ b_3(b_1^2 + b_2^2) = 1 \end{cases}$$

Suppose, without loss of generality, that $b_1 \neq b_3$. Multiply the first equation by b_1, the third by b_3 and take the difference. Then

$$b_2^2(b_1^2 - b_3^2) = b_1 - b_3, \ i.e., \ b_2^2(b_1 + b_3) = 1.$$

Define $x = b_1/b_2$, $y = b_3/b_2$. The above and the second equation imply

$$x^2 + y^2 = 1/b_2^3 = x + y.$$

Now divide the system by b_2^3 and take the difference between the two last equations:

$$x^2 + y^2 = y(x^2 + 1).$$

Hence $x = yx^2$ and so $y = 1/x$. Therefore $x^4 + 1 = x^3 + x$. One easily checks that $x = 1$ is the only positive solution to this equation. Therefore $x = y$ and $b_1 = b_3$, which is absurd. Therefore $b_1 = b_2 = b_3$ and so $a_m = a =: 2^{-1/6}$.

We observe that $\mathbf{v} = (1, 1, 0)u_0$ is also a bound-state. Now compute the action of $\mathbf{u}$ and $\mathbf{v}$:

$$I(\mathbf{u}) = \frac{3}{2^{1/6}} I(u_0) > 2I(u_0) = I(\mathbf{v}).$$

This means that $\mathbf{u}$ cannot be a ground-state, which ends the proof. □

Remark 3.1.1. We conjecture that the previous result applies for more general M and p, with $k_{mm} = \mu, \forall m$, and $k_{jm} = b$, $\forall m \neq j$, $\mu \ll b$. In fact, a necessary and sufficient condition for the existence of fully nontrivial ground-states should be (see equation (3.1.5))

$$\frac{M}{(M-1)^{1/p}} \leqslant \frac{M-1}{(M-2)^{1/p}}.$$

In fact, *if the only possible fully nontrivial ground-state is the one with all components equal*, this condition determines whether it truly is a ground-state. Numerical simulations suggest that this uniqueness should hold for any p, M. We advise the reader to compare this hypothesis with the condition for existence of fully nontrivial ground-states that appears in [52].

3.2 Mandel's characteristic function

Once again, consider system (3.1.1): for a given nonempty symmetric subset P of $\{1, ..., M\}^2$ and $\beta \in \mathbb{R}$,

$$\Delta u_m - \omega_m u_m + \sum_{(j,m) \notin P} k_{jm} |u_j|^{p+1} |u_m|^{p-1} u_m + \sum_{(j,m) \in P} \beta k_{jm} |u_j|^{p+1} |u_m|^{p-1} u_m = 0,$$

with $m = 1, \ldots, M$. For the sake of simplicity, we suppose that $k_{jm} > 0, \forall (j,m) \in P$. We define

$$J_P(\mathbf{u}) = \sum_{(j,m) \in P} k_{jm} \int |u_m|^{p+1} |u_j|^{p+1}, \quad J_{NP}(\mathbf{u}) = \sum_{(j,m) \notin P} k_{jm} \int |u_m|^{p+1} |u_j|^{p+1}.$$

As before, we shall place a subscript β whenever a solution, function or set depends on β. Suppose that $G^*_\beta \neq \emptyset$. Therefore there exists $\mathbf{u}_\beta$ nontrivial bound-state such that

$$I(\mathbf{u}_\beta) \leqslant \mathcal{I}^{sem}_\beta.$$

Since $I(\mathbf{u}_\beta) = J_\beta(\mathbf{u}_\beta)$,

$$(\mathcal{I}^{sem}_\beta)^p \geqslant \frac{I(\mathbf{u}_\beta)^{p+1}}{J_\beta(\mathbf{u}_\beta)}, \text{ i.e. } J_{NP}(\mathbf{u}_\beta) + \beta J_P(\mathbf{u}_\beta) \geqslant I(\mathbf{u})^{p+1}(\mathcal{I}^{sem}_\beta)^{-p}.$$

Hence

$$\beta \geqslant \frac{I(\mathbf{u}_\beta)^{p+1}(\mathcal{I}^{sem}_\beta)^{-p} - J_{NP}(\mathbf{u}_\beta)}{J_P(\mathbf{u}_\beta)} =: B_\beta(\mathbf{u}_\beta).$$

Define

$$\hat\beta = \inf_{\mathbf{u}\in(H^1(\mathbb{R}^d)\backslash\{0\})^M} B_\beta(\mathbf{u}).$$

Then $\beta < \hat\beta$ clearly implies $G^*_\beta = \emptyset$. Moreover, it is not hard to check that, if $\beta > \hat\beta$, $G^*_\beta = G_\beta$. Also, if $\hat\beta = \beta$, $G_\beta\backslash G^*_\beta \neq \emptyset$.

Let us look deeper into the properties of $\hat\beta$. Suppose, for instance, that $\hat\beta_0 > \beta_0$. Then

$$\frac{I(\mathbf{u})^{p+1}}{J_{\hat\beta_0}(\mathbf{u})} > (I^{sem}_{\beta_0})^p, \quad \forall \mathbf{u} : J_P(\mathbf{u}) \neq 0.$$

Take $\beta_0 \leqslant \beta \leqslant \hat\beta_0$. If $\mathbf{u}^{sem}_\beta$ (the best semitrivial bound-state) satisfies $J_P(\mathbf{u}^{sem}_\beta) \neq 0$, then

$$(I^{sem}_\beta)^p = \frac{I(\mathbf{u}^{sem}_\beta)^{p+1}}{J_\beta(\mathbf{u}^{sem}_\beta)} \geqslant \frac{I(\mathbf{u}^{sem}_\beta)^{p+1}}{J_{\hat\beta_0}(\mathbf{u}^{sem}_\beta)} > (I^{sem}_{\beta_0})^p,$$

which is absurd, by the monotonicity properties. Therefore $J_P(\mathbf{u}^{sem}_\beta) = 0$, for all $\beta \in [\beta_0, \hat\beta_0]$. In turn, by the definition of $\mathcal{I}^{sem}_\beta$, we see that it is constant in this interval and so the function $\beta \mapsto \hat\beta$ is constant on $[\beta_0, \hat\beta_0]$. Moreover, since $\beta < \hat\beta_0 = \hat\beta$, $G^+_\beta = \emptyset$ for all $\beta \in [\beta_0, \hat\beta_0)$.

Thus condition $\hat\beta > \beta$ has more implications than the simple dichotomy seen in [56]. Precisely because of this fact, is not as powerful when studying the nonemptiness of G^* as one would desire: for example, if P contains all diagonal terms, one has $\beta \geqslant \hat\beta$ for all β.

Proposition 3.2.1. *Let $\mathbf{u}$ be a semitrivial bound-state for (M-NLS') and suppose that $P \subset \{1, ..., M\}$ is such that*

$$(j, m) \in P \Rightarrow \mathbf{u}_m\mathbf{u}_j \equiv 0. \tag{3.2.1}$$

Then, for β large, $\mathbf{u} \notin G$.

Proof. Suppose that $\mathbf{u}$ is a ground-state for a sequence $\beta_n \to \infty$. The hypothesis (3.2.1) implies that

$$\frac{I(\mathbf{u})^{p+1}}{J_\beta(\mathbf{u})} = \frac{I(\mathbf{u})^{p+1}}{J(\mathbf{u})}.$$

Since, for each β_n, $\mathbf{u}$ is the semitrivial bound-state with the lowest action, this implies that $\hat{\beta}_n = \hat{1}, \forall n$. Taking n_0 large, $\beta_{n_0} > \hat{1} = \hat{\beta}_{n_0}$, which implies that $G = G^*$, contradicting $\mathbf{u} \in G$. ☐

Proposition 3.2.2. *Consider system (M-NLS') and fix $p \geqslant 1$. Suppose that $k_{jm} = \beta > 0$, $\forall m \neq j$, and $k_{mm} = \mu > 0, \forall m$. If $\beta \ll \mu$, any ground-state has exactly one nonzero component.*

Proof. Through a normalization, one may assume $\mu = 1$. From [56], the property is true for $M = 2$. We now proceed by induction: suppose that the result is true for $M-1$ equations. Then there exists β_{M-1} and $\mathbf{u}_0$ with only one nonzero component such that

$$\frac{I(\mathbf{u})^{p+1}}{J_\beta(\mathbf{u})} \geqslant \frac{I(\mathbf{u}_0)^{p+1}}{J_\beta(\mathbf{u}_0)} = \frac{I(\mathbf{u}_0)^{p+1}}{J(\mathbf{u}_0)} = I(\mathbf{u}_0)^p, \forall \mathbf{u} \text{ semitrivial }, \forall 0 < \beta < \beta_{M-1}. \quad (3.2.2)$$

Consider the function $\beta \mapsto \hat{\beta}$. Since $\mathbf{u}_0$ has only one nonzero component, $\hat{\beta}$ is constant on $(0, \beta_{M-1})$. Take any $\mathbf{u}$ such that $J_P(\mathbf{u}) \neq 0$. W.l.o.g., assume that the last component has the largest L^{2p+2} norm. For each $1 \leqslant m \leqslant M$, define

$$r_m = \frac{\|u_m\|_{2p+2}}{\|u_M\|_{2p+2}} \leqslant 1,\ V_m = ((v_m)_1, ..., (v_m)_M),\ (v_m)_j = u_m \delta_{jm}$$

Then, using (3.2.2) and $J_\beta(V_m) = J(V_m)$,

$$\frac{I(\mathbf{u})^{p+1}(I(\mathbf{u}_0))^{-p} - J_{NP}(\mathbf{u})}{J_P(\mathbf{u})} = \frac{\left(\sum_{m=1}^M I(V_m)\right)^{p+1} I^{-p}(\mathbf{u}_0) - \sum_{m=1}^M J(V_m)}{J_P(\mathbf{u})}$$

$$= \frac{\left(\frac{I(V_M)}{J(V_M)^{1/(p+1)}} + \sum_{m=1}^{M-1} r_m^2 \frac{I(V_m)}{J(V_m)^{1/(p+1)}}\right)^{p+1} I^{-p}(\mathbf{u}_0) - 1 - \sum_{m=1}^{M-1} r_m^{2p+2}}{\frac{J_P(\mathbf{u})}{J(V_M)}}$$

$$\geqslant \frac{(1 + \sum_{m=1}^{M-1} r_m^2)^{p+1} - 1 - \sum_{m=1}^{M-1} r_m^{2p+2}}{2\sum_{m=1}^{M-1} r_m^{p+1} + \sum_{j,m=1}^{M-1} r_m^{p+1} r_j^{p+1}} =: g(r_1, ..., r_{M-1})$$

Since $p \geqslant 1$, g is bounded below over the set $[0,1]^{M-1}$ by a constant $m > 0$. Hence $B_\beta(\mathbf{u}) \geqslant m > 0, \forall \mathbf{u}$. Therefore, taking $\beta_M = \min\{\beta_{M-1}, m\}$, we see that $\hat{\beta} \geqslant m > \beta$, for $0 < \beta < \beta_M$. The properties of $\hat{\beta}$ imply that the result is true for M equations. ☐

3.3 The cubic nonlinearity case

In this section, to simplify some notations and to highlight the different roles each family of parameters has, we define

$$\mu_m := k_{mm}, \quad b_{jm} = k_{jm},\ m \neq j, \quad j, m = 1, \dots M.$$

Using this notation, system (M-NLS') becomes

$$\Delta u_m - \omega_m u_m + \mu_m |u_m|^2 u_m + \sum_{j=1, j\neq m}^{M} b_{jm} |u_j|^2 u_m = 0, \quad m = 1, ..., M.$$

Moreover, we shall write

$$|u|_\omega^2 = \omega \|u\|_2^2 + \|\nabla u\|_2^2.$$

Throughout this section, we shall assume that $\mu_m, b_{jm} \geqslant 0$, for all j, m.

3.3.1 Existence of fully nontrivial ground states

In this section, we prove three results that guarantee the existence of fully nontrivial ground-states. Before we proceed, we need the following definition:

Definition 3.3.1. *Let $\alpha > 1$ and $k \geqslant 2$. We say that a vector $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ is α-admissible ($\mathbf{a} \in \mathcal{A}_\alpha$) if*

$$\max_{1\leqslant m\leqslant k} a_m < \alpha \min_{1\leqslant m\leqslant k} a_m.$$

The next lemma ensures that, up to rotations, all ground-states are nonnegative. Since, in terms of nontriviality, rotations are harmless, we may restrict ourselves to the case where all components of the ground-states are nonnegative. In that case, we may rewrite our system as

$$\Delta u_m - \omega_m u_m + \mu_m u_m^3 + \sum_{j=1, j\neq m}^{M} b_{jm} u_j^2 u_m = 0, \quad m = 1, ..., M. \tag{3.3.1}$$

Lemma 3.3.2. *Suppose that $\mu_m, b_{jm} \geqslant 0$, for all $1 \leqslant m, j \leqslant M$. If $\mathbf{u} = (u_1, \dots, u_M) \in G$, then there exists $\theta_j \in \mathbb{R}$ such that*

$$u_j = e^{i\theta_j} |u_j|, \quad 1 \leqslant j \leqslant M.$$

Proof. Since $\mathbf{u} \in G$, it is easy to check that $\mathbf{v} = (|u_1|, \dots, |u_M|)$ is also in G. It then follows from a standard application of the maximum principle that either $u_j \equiv 0$ or $|u_j(x)| > 0$ for any $x \in \mathbb{R}^d$ and any j. We now conclude as in the proof of Theorem 2.2.4. □

Theorem 3.3.3 (Existence Result I)**.** *Let $M \geqslant 3$, $0 < \omega_1, \cdots, \omega_M$ and $b_{jm} \equiv b > 0$, with*

$$(\omega_1, \cdots, \omega_M) \in \mathcal{A}_{1+\frac{1}{M-2}}.$$

Then there exists a constant $B = B(\omega_m, \mu_m) > 0$, such that, for $b > B$, $G = G^*$.

Proof. We will proceed by mathematical induction on the number of equations. As mentioned in the introduction, it is well-known that the result holds true for $M = 2$ equations. Indeed, in this case, for all $\omega_1, \omega_2 > 0$, the system (3.3.1) admits a fully nontrivial ground state for b large enough.

We now consider an admissible $\omega = (\omega_1, \ldots, \omega_M) \in \mathcal{A}_{\alpha(M)}$, with

$$\alpha(M) = 1 + \frac{1}{M-2}.$$

Given $X \subsetneq \{1, 2, \ldots, M\}$, we denote by $\mathcal{I}^X$ the ground state level of the system

$$\Delta u_m - \omega_m u_m + \mu_m u_m^3 + b u_m \sum_{j \in X, j \neq m} u_j^2 = 0, \qquad m \in X.$$

Notice that if $X \subset \{1, \ldots, M\}$, then $(\omega_m : m \in X) \in \mathcal{A}_{\alpha(\#X)}$. Hence, following the ideas in [62], we assume, by induction hypothesis, that there exists a ground state level $\mathcal{I}^X$ with $\#X = M-1$ and, for any Y with $\#Y < M-1$, $\mathcal{I}^X < \mathcal{I}^Y$. Without loss of generality, we assume that

$$\mathcal{I}^{sem} := \mathcal{I}^{\{1,\ldots,M-1\}} = \min\{\mathcal{I}^X : \#X = M-1\},$$

where $\mathcal{I}^{sem}$ is achieved by the fully nontrivial ground state $(u_1, \ldots, u_{M-1})$, solution of

$$\Delta u_m - \omega_m u_m + \mu_m u_m^3 + b u_m \sum_{\substack{j=1 \\ j \neq m}}^{M-1} u_j^2 = 0, \qquad m = 1, \ldots, M-1.$$

Define $\mathcal{N}$ as in Proposition 2.1.7. We prove our result by exhibiting $\mathbf{v} = (v_1, \ldots, v_M) \in \mathcal{N}$, $v_m \not\equiv 0$, such that $I(\mathbf{v}) < I(u_1, \ldots, u_{M-1}, 0) = \mathcal{I}^{sem}$, which guarantees that the energy level of $\mathbf{v}$ is inferior to the energy level of any solution of (3.3.1) with trivial components.
For a fixed $w \in H^1(\mathbb{R}^d)$, $w \neq 0$, and $\theta > 0$, we choose $t > 0$ such that

$$\mathbf{v} = (tu_1, \ldots, tu_{M-1}, t\theta w) \in \mathcal{N}.$$

A straightforward computation leads to

$$t^2 = \frac{1 + \theta^2 C_1}{1 + \mu_M \theta^4 C_2 + 2b \sum_{m=1}^{M-1} \theta^2 D_m},$$

where

$$C_1 = \frac{|w|^2_{\omega_M}}{\sum_{m=1}^{M-1} |u_m|^2_{\omega_m}}, \quad C_2 = \frac{\|w\|_4^4}{\sum_{m=1}^{M-1} |u_m|^2_{\omega_m}} \text{ and } D_m = \frac{\|u_m w\|_2^2}{\sum_{m=1}^{M-1} |u_m|^2_{\omega_m}}.$$

Now, since $(tu_1, \ldots, tu_{M-1}, t\theta w) \in \mathcal{N}$,

$$I(tu_1, \ldots, tu_{M-1}, t\theta w) = \frac{1}{4}\Big(\sum_{m=1}^{M-1} |tu_m|^2_{\omega_m} + \theta^2 |t\theta w|^2_{\omega_M} \Big)$$

$$= \frac{t^2}{4}\Big(1 + C_1\theta^2\Big) \sum_{m=1}^{M-1} |u_m|^2_{\omega_m}.$$

Thus

$$I(tu_1, \dots, tu_{M-1}, t\theta w) < \frac{1}{4} \sum_{m=1}^{M-1} |u_m|^2_{\omega_m} = I_d(u_1, \dots, u_{M-1}, 0)$$

if and only if

$$\frac{(1+\theta^2 C_1)^2 - 1 - \mu_M \theta^4 C_2}{\theta^2} < 2b \sum_{m=1}^{M-1} D_m.$$

By taking the limit $\theta \to 0^+$ we conclude that we need to exhibit w such that $C_1 < b\sum_{m=1}^{M-1} D_m$, that is

$$|w|^2_{\omega_M} < b \sum_{m=1}^{M-1} \|u_m w\|_2^2. \tag{3.3.2}$$

This is straightforward if there exists $1 \leqslant m_0 \leqslant M-1$ such that $\omega_M \leqslant \omega_{m_0}$. Indeed, in this case, by multiplying the equation

$$\Delta u_{m_0} - \omega_{m_0} u_{m_0} + \mu_{m_0} u_{m_0}^3 + b u_{m_0} \sum_{\substack{m=1\\ m\neq m_0}}^{M-1} u_m^2 = 0$$

by u_{m_0} and integrating, we obtain, for $b > \max\{\mu_m \,:\, 1 \leqslant i \leqslant M\}$,

$$|u_{m_0}|^2_{\omega_M} \leqslant |u_{m_0}|^2_{\omega_{m_0}} = \mu_{m_0}\|u_{m_0}\|_4^4 + b \sum_{\substack{m=1\\ m\neq m_0}}^{M-1} \|u_m u_{m_0}\|_2^2 < b \sum_{m=1}^{M-1} \|u_m u_{m_0}\|_2^2$$

and we can choose $w = u_{m_0}$. Hence, in the rest of the this proof, we may assume that $\omega_M > \omega_m$ for all $1 \leqslant i \leqslant M-1$.
Without loss of generality, we may also assume that

$$\|u_1\|_4 \geqslant \|u_m\|_4 \text{ for all } 1 \leqslant j \leqslant M-1. \tag{3.3.3}$$

We then choose $w = u_1$ and, since

$$|u_1|^2_{\omega_1} = \mu_1\|u_1\|_4^4 + b \sum_{m=2}^{M-1} \|u_m u_1\|_2^2, \tag{3.3.4}$$

the condition (3.3.2) is equivalent to

$$\omega_M - \omega_1 < (b - \mu_1)\frac{\|u_1\|_4^4}{\|u_1\|_2^2}.$$

By (3.3.4),

$$\omega_1 < \mu_1 \frac{\|u_1\|_4^4}{\|u_1\|_2^2} + b \sum_{m=2}^{M-1} \frac{\|u_1 u_m\|_2^2}{\|u_1\|_2^2} < \mu_1 \frac{\|u_1\|_4^4}{\|u_1\|_2^2} + \frac{b}{2} \sum_{m=2}^{M-1} \frac{\|u_1\|_4^4 + \|u_m\|_4^4}{\|u_1\|_2^2}$$

$$< ((M-2)b+\mu_1)\frac{\|u_1\|_4^4}{\|u_1\|_2^2},$$

where we have used the hypothesis (3.3.3).
Hence, for (3.3.2) to hold, it is sufficient that $\omega_M - \omega_1 < \dfrac{b-\mu_1}{(M-2)b+\mu_1}\omega_1$.
By taking the limit $b \to +\infty$, this condition holds for large b if $\omega_M < (1+\frac{1}{M-2})\omega_1$, which is true if $(\omega_1, \ldots, \omega_M) \in \mathcal{A}_{1+\frac{1}{M-2}}$. □

In the cubic nonlinearity case, if $b_{jm} \equiv b > 0$, one is able to extend Theorem 2.2.4 to system (M-NLS'). The following result states that, if k components have the same ω, then any ground state has those components proportional to each other.

Theorem 3.3.4. *Let $M \geqslant 3$, $b_{jm} \equiv b > 0$ and $0 < \omega_1, \ldots, \omega_M$. Suppose that, for some $k \in \{2, \ldots, M\}$, $\omega_1 = \cdots = \omega_k \equiv \omega$. Consider the function $f : \mathbb{R}^k \mapsto \mathbb{R}$ defined by*

$$f(y_1, \cdots, y_k) = \sum_{\substack{j,m=1\\ m\neq j}}^{k} b y_m^2 y_j^2 + \sum_{m=1}^{k} \mu_m y_m^4,$$

set $f_{max} = \max\limits_{|\mathbf{y}|=1} f(\mathbf{y})$ and define the set $\mathcal{Y}$ in the following way:

1. *if $\max\{\mu_1, \ldots, \mu_k\} > b$, setting e_m to be the m-th vector of the canonical basis of $\mathbb{R}^k$,*
$$\mathcal{Y} = \left\{ \pm e_m : \ i \text{ is such that } \mu_m = \max_{j=1,\ldots,k} \mu_j \right\};$$
2. *if $\max\{\mu_1, \ldots, \mu_k\} < b$, then $f_{max} < b$ and*
$$\mathcal{Y} = \left\{ \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k : y_m = \pm \left(\frac{f_{max} - b}{\mu_m - b} \right)^{1/2} \right\};$$
3. *if $\max\{\mu_1, \ldots, \mu_k\} = b$,*
$$\mathcal{Y} = \left\{ \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k : \ |\mathbf{y}| = 1 \text{ and } y_m = 0 \ \forall m \text{ such that } \mu_m < b \right\}.$$

Then, denoting by G_{M-k+1} the set of ground states for the system

$$\begin{cases} \Delta u - \omega u + \mu u^3 + bu \sum\limits_{j>k} u_j^2 = 0 \\ \Delta u_m - \omega_m u_m + \mu_m u_m^3 + b u^2 u_m + b u_m \sum\limits_{\substack{j>k\\ j\neq m}} u_j^2 = 0, \ \ m = k+1, \ldots, M \end{cases}$$

with $\mu = f_{max}$, one has

$$G = \{(u\mathbf{y}, u_{k+1}, \ldots, u_M) \in (H^1(\mathbb{R}^d))^M : (u, u_{k+1}, \ldots, u_M) \in G_{M-k+1}, \mathbf{y} \in \mathcal{Y}\}.$$

Proof. We proceed in two steps.
Step 1. Characterization of ground states. Take $\mathbf{u} = (u_1, \cdots, u_M) \in G$, and let us show that $\mathbf{u} = (u\mathbf{y}, u_{k+1}, \ldots, u_M)$, where $(u, u_{k+1}, \ldots, u_M) \in G_{M-k+1}$ and $\mathbf{y} \in \mathcal{Y}$. Define

$$u(x) = \left(\sum_{m=1}^{k} u_m^2(x) \right)^{1/2}.$$

If $u = 0$, there is nothing left to prove. Otherwise, let $\mathcal{Y}$ be the (nonempty) set of solutions to the maximization problem

$$f(\mathbf{y}_0) = f_{max} = \max_{|\mathbf{y}|=1} f(\mathbf{y}), \quad |\mathbf{y}_0| = 1. \tag{3.3.5}$$

Take $\mathbf{y} \in \mathcal{Y}$ and $\mathbf{w} = (w_1, \ldots, w_M) = (u\mathbf{y}, u_{k+1}, \ldots, u_M)$. Let us show that $\mathbf{w}$ is also a ground state solution. We have

$$\begin{aligned}
J(\mathbf{u}) &= \sum_{m=1}^{M} \mu_m \|u_m\|_4^4 + \sum_{j \neq m} b\|u_m u_j\|_2^2 = \int f(|u_1|, \cdots, |u_{M-1}|) + \sum_{i>k} \int \mu_m u_m^4 \\
&\quad + \sum_{j,m>k, m\neq j} \int b u_m^2 u_j^2 + \sum_{i>k} \sum_{j=1}^{k} \int 2b u_m^2 u_j^2 \\
&= \sum_{i>k} \int \mu_m u_m^4 + b \sum_{j,m>k,m\neq j} \int u_m^2 u_j^2 + 2b \sum_{i>k} \int u_m^2 u^2 + \int f\left(\frac{|u_1|}{u}, \cdots, \frac{|u_{M-1}|}{u} \right) u^4 \\
&\leqslant \sum_{i>k} \int \mu_m u_m^4 + b \sum_{j,m>k,m\neq j} \int u_m^2 u_j^2 + 2b \sum_{i>k} \int u_m^2 |\mathbf{y}|^2 u^2 + \int f(\mathbf{y}) u^4 \\
&= \sum_{m=1}^{M} \mu_m \|w_m\|_4^4 + \sum_{j\neq m} b\|w_m w_j\|_2^2 = J(\mathbf{w})
\end{aligned}$$

Furthermore, since $\|\nabla \mathbf{u}\|_2^2 \leqslant \sum_{m=1}^{k} \|\nabla u_m\|_2^2$,

$$\begin{aligned}
I(\mathbf{w}) &= \sum_{m=1}^{M} |w_m|_{\omega_m}^2 = \sum_{m=k+1} |u_m|_{\omega_m}^2 + \sum_{m=1}^{k} |u\mathbf{y}_m|_{\omega}^2 \\
&= \sum_{m=k+1}^{M} |u_m|_{\omega_m}^2 + |u|_{\omega}^2 \leqslant \sum_{m=1}^{k} |u_m|_{\omega_m}^2 = I(\mathbf{u}).
\end{aligned}$$

By Lemmata 2.1.2 and 2.1.3, $\mathbf{w}$ is a ground state. Moreover,

$$f\left(\frac{|u_1|}{u}, \ldots, \frac{|u_k|}{u} \right) = f(\mathbf{y}) \qquad \text{for a.e. } x \in \mathbb{R}^d,$$

so that, if we write $\mathbf{u}(x) = (u(x)\mathbf{z}(x), u_k(x), \ldots, u_M(x))$, with $\mathbf{z}_m = |u_m|/u$, then $\mathbf{z}(x) \in \mathcal{Y}$ for a.e. $x \in \mathbb{R}^d$. Observe that $\mathbf{z} \in C^\infty$ as $\mathbf{u}$ and u are both smooth and $u \neq 0$.

Let us now check that $(u, u_{k+1}, \ldots, u_M) \in G_{M-k+1}$. Since $\mathbf{y}$ is a solution to the maximization problem (3.3.5), there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$\mu y_m = \mu_m y_m^3 + b \sum_{j=1, j\neq m}^{k} y_j^2 y_m, \; m = 1, \ldots, k.$$

Multiplying by $\mathbf{y}_m$ and summing in i, we obtain $\mu = f_{max}$. Hence, since $\mathbf{w}$ is a solution of (3.3.1), the pair $(u, u_{k+1}, \ldots, u_M)$ must be a solution of

$$\begin{cases} \Delta u - \omega u + \mu u^3 + bu \sum\limits_{j>k} u_j^2 = 0 \\ \Delta u_m - \omega_m u_m + \mu_m u_m^3 + bu^2 u_m + bu_m \sum_{j>k \, j\neq m} bu_j^2 = 0, \; m = k+1, \ldots, M. \end{cases}$$

Moreover, the minimality of $\mathbf{w}$ implies that $(u, u_{k+1}, \ldots, u_M) \in G_{M-k+1}$.

Finally, let us see that actually $\mathbf{z}$ is constant. We know that $\mathbf{u} = (u\mathbf{z}, u_{k+1}, \ldots, u_d)$ is also a sollution of (3.3.1). Inserting this expression onto system (3.3.1) and using the above equations, we see that

$$u\Delta z_m + 2\nabla z_m \cdot \nabla u = 0, \; m = 1, \ldots, k.$$

Arguing as in the proof of Theorem 2.2.4 this implies that

$$\int |\nabla z_m|^2 |u|^2 = 0, \quad m = 1, \ldots, k.$$

Since $u > 0$, we conclude that $\mathbf{z}$ is constant. Therefore

$$G \subseteq \{(u\mathbf{y}, u_{k+1}, \ldots, u_M) \in (H^1(\mathbb{R}^d))^M : (u, u_{k+1}, \ldots, u_M) \in G_{M-k+1}, \mathbf{y} \in \mathcal{Y}\}.$$

The other inclusion comes from the fact that, given $\mathbf{y} \in \mathcal{X}$, $\mu = f_{max}$ and

$$(u, u_{k+1}, \ldots, u_M) \in G_{M-k+1},$$

then $(u\mathbf{y}, u_{k+1}, \ldots, u_M)$ is a bound state of (3.3.1) with action equal to that of $\mathbf{u}$.

Step 2. Expression of $\mathcal{Y}$. Take $\mathbf{y} = (y_1, \ldots, y_k)$ such that $|\mathbf{y}| = 1$. Then

$$f(\mathbf{y}) = \sum_{m=1}^{k} \mu_m y_m^4 + \sum_{m=1}^{k} by_m^2(1 - y_m^2) = b + \sum_{m=1}^{k} y_m^4(\mu_m - b).$$

Define $g(z_1, \cdots, z_k) = b + \sum_{m=1}^{k} z_m^2(\mu_m - b)$. Then $\mathbf{y}$ is a maximizer of f on the unit ball if and only if $\mathbf{z} = (y_1^2, \cdots, y_k^2)$ is a maximizer of g on the convex set $\Delta_k = \{\mathbf{z} \in \mathbb{R}^k : z_m \geqslant 0, \; \sum_m z_m = 1\}$. Now one must split in several cases:

- If $\max_m \mu_m > b$, writing $\Delta_k^+ = \{\mathbf{z} \in \mathbb{R}^k : \sum_m z_m = 1; z_m = 0, \forall m : \mu_m \leqslant b\}$,

$$\max_{\mathbf{z}\in\Delta_k} g(\mathbf{z}) < \max_{\mathbf{z}\in\Delta_k^+} g(\mathbf{z})$$

  Since g is a strictly convex function on Δ_k^+, its maximum is attained at some vertex. Hence the maximizers are e_m, for i's such that $\mu_m = \max\{\mu_1, \ldots, \mu_k\}$.

- If $\max_m \mu_m < b$, then g is a strictly concave function and so the maximum is attained at a unique point $\mathbf{z}_0 = ((z_0)_1, \cdots, (z_0)_k)$ on the interior of Δ_k. This implies that, for some Lagrange multiplier $\eta \in \mathbb{R}$,

$$(z_0)_m(\mu_m - b) = \eta \qquad \text{for every } m = 1, \ldots, k.$$

Multiplying the i-th equation by $(z_0)_m$ and summing up, we obtain $\eta = f_{max} - b$ and therefore

$$(z_0)_m = \frac{f_{max} - b}{\mu_m - b};$$

- Finally, if $\max_m \mu_m = b$, writing $\Delta_k^0 = \{\mathbf{z} \in \mathbb{R}^k : \sum_m z_m = 1; z_m = 0, \forall m : \mu_m < b\}$,

$$\max_{\mathbf{z}\in\Delta_k} g(\mathbf{z}) \leqslant \max_{\mathbf{z}\in\Delta_k^0} g(\mathbf{z}).$$

Since g is constant on Δ_{M-1}^0, we obtain the desired expression. $\square$

We now prove a more refined version of Corollary 3.1.8:

Theorem 3.3.5 (Existence Result II). *Let $M \geqslant 3$, $0 < \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_M$ and $b_{jm} \equiv b$. Then, setting $\omega = \omega_2/\omega_1$, $\rho(M) = (M-2)/(M-1)$ and*

$$\alpha = \alpha(\omega_1/\omega_2, M, d) := \left(1 - \frac{\rho(M) - \rho(M-1)}{\sqrt{2\omega^2 \frac{(\rho(M-1)+\omega)^2+\omega^2}{(\rho(M-1)+2\omega)^2} + \rho(M)}}\right)^{-\frac{2}{4-d}},$$

if

$$(\omega_2, \ldots, \omega_M) \in \mathcal{A}_\alpha,$$

there exists a constant $B = B(\omega_m, \mu_m) > 0$, such that, for $b > B$, $G = G^*$.

Proof. Along this proof, we denote by $\mathcal{I}(\omega_1, \omega_2, \ldots, \omega_M)$ the ground state action level of system (3.3.1), and by $\mathcal{I}^{sem}(\omega_1, \omega_2, \ldots, \omega_M)$ the semitrivial ground state level, that is

$$\mathcal{I}^{sem}(\omega_1, \omega_2, \ldots, \omega_M) = \min\{\mathcal{I}^X : X \subset \{1, \ldots, M\}, \ \#X \leqslant M-1\}.$$

Our aim is to prove that

$$\mathcal{I}(\omega_1, \omega_2, \ldots, \omega_M) < \mathcal{I}^{sem}(\omega_1, \omega_2, \ldots, \omega_M). \tag{3.3.6}$$

Since the proof is long, we divide it into several steps.

Step 1. It is enough to prove (3.3.6) in the case $\mu_m = 0$ for all i and $b = 1$. In fact, by considering the scaling $U_m = \sqrt{b}u_m$ and by the continuity of the levels $\mathcal{I}$ and $\mathcal{I}^{sem}$ (cf. Proposition 3.1.4), one may consider the case b large as a perturbation of this case. Also, using the scaling $U_m(x) = \omega_1^{-1/2}u_m(\omega_1^{-1/2}x)$, the vector $(\omega_1, \omega_2, \ldots, \omega_M)$ becomes $(1, \omega_2/\omega_1, \ldots, \omega_M/\omega_1)$. Hence, one may focus on the case $\omega_1 = 1$, to simplify the notations.

Step 2. Let us now start with the core of the proof. From Lemma 3.1.2, one has the following properties:

$$\mathcal{I}(1,\omega_2,\dots,\omega_M) \leqslant \mathcal{I}(1,\omega_M,\dots,\omega_M) \tag{3.3.7}$$

$$\mathcal{I}^{sem}(1,\omega_2,\dots,\omega_M) \geqslant \mathcal{I}^{sem}(1,\omega_2,\dots,\omega_2) \tag{3.3.8}$$

and, using a suitable scaling,

$$\begin{aligned}\mathcal{I}(1,\omega_M,\dots,\omega_M) &= \left(\frac{\omega_M}{\omega_2}\right)^{\frac{4-d}{2}} \mathcal{I}\left(\frac{\omega_2}{\omega_M},\omega_2,\dots,\omega_2\right) \\ &\leqslant \left(\frac{\omega_M}{\omega_2}\right)^{\frac{4-d}{2}} \mathcal{I}(1,\omega_2,\dots,\omega_2).\end{aligned} \tag{3.3.9}$$

We will now focus our attention on $\mathcal{I}(1,\omega_2,\dots,\omega_2)$, proving that

$$\mathcal{I}(1,\omega_2,\dots,\omega_2) < K\mathcal{I}^{sem}(1,\omega_2,\dots,\omega_2), \tag{3.3.10}$$

for some constant $K>0$.

Step 3. Now take a semitrivial $\mathbf{u}=(u_1,\dots,u_M)$ that achieves $\mathcal{I}^{sem}(1,\omega_2,\dots,\omega_2)$. Since $\omega_1=\min\{\omega_1,\dots,\omega_M\}$, using an argument similar to that in the proof of Theorem 3.3.3, one has $u_1\neq 0$. Since one may order the remaining components as one wishes, we shall suppose that $u_M=0$. Next, we want to apply Theorem 3.3.4. To that end, following the notations of that theorem, we determine explicitly f_{max} and $\mathcal{Y}$. First of all, since $\mu_m=0$, one has

$$\mathcal{Y}=\left\{\mathbf{y}=(x_2,\dots,x_{M-1})\in\mathbb{R}^{M-2} : x_m=\pm\left(1-f_{max}\right)^{1/2}\right\}.$$

On the other hand, since $\mathcal{Y}$ is the set of solutions of the maximization problem

$$f(\mathbf{y}_0)=f_{max}=\max_{|\mathbf{y}|=1} f(\mathbf{y}), \quad |\mathbf{y}_0|=1,$$

any element in $\mathcal{Y}$ has unit norm. Since

$$\left((1-f_{max})^{1/2},\dots,(1-f_{max})^{1/2}\right)\in\mathcal{Y},$$

we obtain $(1-f_{max})^{1/2}=(M-2)^{-1/2}$, that is, $f_{max}=1-1/(M-2)$. Theorem 3.3.4 now implies that

$$\mathbf{u}=(u_1,\pm(M-2)^{-1/2}u,\dots,\pm(M-2)^{-1/2}u,0)$$

where (u_1,u) is a (nontrivial) ground-state of

$$\begin{cases}\Delta w_1 - w_1 + w_2^2 w_1 = 0\\ \Delta w_2 - \omega_2 w_2 + \mu w_2^3 + w_1^2 w_2 = 0\end{cases}, \quad \mu = 1-\frac{1}{M-2}. \tag{3.3.11}$$

Step 4. We will provide an estimate of the L^4 norm of u_1 in terms of the L^4 norm of u. To this end, consider

$$\mathbf{w} = (w_1, w_2) = \Big(u_1, \frac{1}{\sqrt{\omega_2}} u_1\Big).$$

Since $\omega_2 \geqslant 1$, a simple computation yields

$$|w_1|_1^2 + |w_2|_{\omega_2}^2 = |u_1|_1^2 + \frac{1}{\omega_2}|u_1|_{\omega_2}^2 \leqslant 2|u_1|_1^2. \tag{3.3.12}$$

Furthermore, since (u_1, u) is in particular a solution of (3.3.11), we obtain that

$$|u_1|_1^2 = \int u_1^2 u^2 = |u|_{\omega_2}^2 - \mu \int u^4 \leqslant |u|_{\omega_2}^2.$$

Combining this with (3.3.12), we get $|w_1|_1^2 + |w_2|_{\omega_2}^2 \leqslant |u_1|_1^2 + |u|_{\omega_2}^2$. Now it is easy to see that

$$\mu \int w_2^4 + 2 \int w_1^2 w_2^2 = \frac{\mu + 2\omega_2}{\omega_2^2} \int u_1^4 \leqslant \mu \int u^4 + 2 \int u_1^2 u^2. \tag{3.3.13}$$

Indeed, if the converse inequality was true, (u_1, u) would not be a solution of (2.1.2) and so it would not be a ground-state of (3.3.11).

Now, from (3.3.13), for any $\varepsilon > 0$,

$$\frac{\mu + 2\omega_2}{\omega_2^2} \int u_1^4 \leqslant \mu \int u^4 + \varepsilon \int u_1^4 + \frac{1}{\varepsilon} \int u^4.$$

Choosing $\varepsilon = \dfrac{\mu + 2\omega_2}{2\omega_2^2}$ one obtains the estimate

$$\int u_1^4 \leqslant 2\omega_2^2 \frac{(\mu + \omega_2)^2 + \omega_2^2}{(\mu + 2\omega_2)^2} \int u^4,$$

as wanted.

Step 5. We are now ready to prove (3.3.10). Consider

$$\mathbf{v} = (v_1, \dots, v_M) = (u_1, (M-1)^{-1/2} u, \dots, (M-1)^{-1/2} u).$$

Then

$$|u_1|_1^2 + \sum_{m=2}^{M} |u_m|_{\omega_2}^2 = |v_1|_1^2 + \sum_{m=2}^{M} |v_m|_{\omega_2}^2$$

and

$$\sum_{\substack{j,m=1 \\ j \neq m}}^{M} \|v_m v_j\|_2^2 - \sum_{\substack{j,m=1 \\ j \neq m}}^{M} \|u_m u_j\|_2^2 = \Bigg(2 \sum_{j=2}^{M} \|v_1 v_j\|_2^2 - 2 \sum_{j=2}^{M} \|u_1 u_j\|_2^2\Bigg)$$

$$+\left(\sum_{\substack{j,m=2\\ j\neq m}}^{M}\|v_m v_j\|_2^2-\sum_{\substack{j,m=2\\ j\neq m}}^{M}\|u_m u_j\|_2^2\right)=\left(\frac{M-2}{M-1}-\frac{M-3}{M-2}\right)\|u\|_4^4>0.$$

If one defines

$$t^2=\frac{\sum_{m=1}^{M}|v_m|_{\omega_m}^2}{\sum_{\substack{j,m=1\\ j\neq m}}^{M}\|v_m v_j\|_2^2}$$

then $t\mathbf{v}\in\mathcal{N}(1,\omega_2,\ldots,\omega_2)$ (the Nehari manifold associated with the ground-state level $\mathcal{I}(1,\omega_2,\ldots,\omega_2)$) and

$$\begin{aligned}
t^2&=\frac{\sum_{m=1}^{M}|v_m|_{\omega_m}^2}{\sum_{\substack{j,m=1\\ j\neq m}}^{M}\|v_m v_j\|_2^2}=\frac{\sum_{m=1}^{M}|u_m|_{\omega_m}^2}{\left(\frac{M-2}{M-1}-\frac{M-3}{M-2}\right)\|u\|_4^4+\sum_{\substack{j,m=1\\ j\neq m}}^{M}\|u_m u_j\|_2^2}\\
&=1-\frac{\left(\frac{M-2}{M-1}-\frac{M-3}{M-2}\right)\|u\|_4^4}{\left(\frac{M-2}{M-1}-\frac{M-3}{M-2}\right)\|u\|_4^4+\sum_{\substack{j,m=1\\ j\neq m}}^{M}\|u_m u_j\|_2^2}\\
&=1-\frac{\frac{M-2}{M-1}-\frac{M-3}{M-2}}{\frac{\|u_1 u\|_2^2}{\|u\|_4^4}+\frac{M-2}{M-1}}\leqslant 1-\frac{\frac{M-2}{M-1}-\frac{M-3}{M-2}}{\frac{\|u_1\|_4^2}{\|u\|_4^2}+\frac{M-2}{M-1}}\\
&\leqslant 1-\frac{\frac{M-2}{M-1}-\frac{M-3}{M-2}}{\sqrt{2\omega_2^2\frac{(\mu+\omega_2)^2+\omega_2^2}{(\mu+2\omega_2)^2}}+\frac{M-2}{M-1}}=:C(\omega_2,M)^2
\end{aligned}$$

Hence

$$\mathcal{I}(1,\omega_2,\ldots,\omega_2)\leqslant I(t\mathbf{v})=t^2 I(\mathbf{u})=t^2\mathcal{I}^{sem}(1,\omega_2,\ldots,\omega_2). \tag{3.3.14}$$

Step 6. Putting together all the relations between the various action levels, namely (3.3.7), (3.3.8), (3.3.9) and (3.3.14), one arrives to

$$\begin{aligned}
\mathcal{I}(1,\omega_2,\ldots,\omega_M)&\leqslant t^2\left(\frac{\omega_M}{\omega_2}\right)^{\frac{4-d}{2}}\mathcal{I}^{sem}(1,\omega_2,\ldots,\omega_M)\\
&\leqslant C^2\left(\frac{\omega_M}{\omega_2}\right)^{\frac{4-d}{2}}\mathcal{I}^{sem}(1,\omega_2,\ldots,\omega_M)<\mathcal{I}^{sem}(1,\omega_2,\ldots,\omega_M),
\end{aligned}$$

provided that $\omega_d < \alpha\omega_2$, with $\alpha = C^{-\frac{4}{4-d}}$. $\square$

Finally, we close this section with an existence result for the situation where the interaction coefficients b_{jm} do not necessarily coincide.

Theorem 3.3.6 (Existence Result III)**.** *Consider the system* (3.3.1) *with* $M \geqslant 3$, $\omega = \omega_1 = \cdots = \omega_M$ *and* $b_{jm} = b_{mj} > 0$. *Suppose that*

$$\alpha := \min_m \left(\min_j b_{jm} - \mu_m \right) > 0$$

and

$$\max_{\substack{1 \leqslant i \leqslant M \\ k \neq j}} |b_{jm} - b_{ik}| < \frac{\alpha}{M-2}.$$

Then $G = G^*$.

Proof. Once again, we will proceed by mathematical induction on the number of equations. Following the steps of the proof of Theorem 3.3.3, in order to prove this result we only need to exhibit $w \in H^1(\mathbb{R}^d)$ such that

$$|w|_\omega^2 < \sum_{m=1}^{M-1} b_{iM} \int u_m^2 w^2 \tag{3.3.15}$$

which is the analogous of condition (3.3.2) for the system at hand. Here, $(u_1, \cdots u_{M-1})$, assumed by induction hypothesis to be fully nontrivial, is a ground state achieving the level $\mathcal{I}^{sem} = \mathcal{I}^{\{1,2,\ldots,M-1\}}$ with $\mathcal{I}^{sem} \leqslant \mathcal{I}^X$ for any $\#X \leqslant M-1$.
Without loss of generality, we may assume once again that for all $1 \leqslant i \leqslant M-1$, $\|u_1\|_4 \geqslant \|u_m\|_4$. Multiplying

$$\Delta u_1 - \omega u_1 + \mu_1 |u_1|^2 u_1 + \sum_{j=2}^{M-1} b_{1j} u_j^2 u_1 = 0$$

by u_1 and integrating by parts leads to

$$|u_1|_\omega^2 = \mu_1 \|u_1\|_4^4 + \sum_{j=2}^{M-1} b_{1j} \int u_j^2 u_1^2.$$

Hence, taking $w = u_1$, condition (3.3.15) is equivalent to

$$\mu_1 \|u_1\|_4^4 + \sum_{j=2}^{M-1} b_{1j} \int u_j^2 u_1^2 < \sum_{m=1}^{M-1} b_{iM} \int u_m^2 u_1^2,$$

that is

$$\sum_{j=2}^{M-1} (b_{jM} - b_{1j}) \int u_j^2 u_1^2 + (b_{1M} - \mu_1) \int u_1^4 > 0.$$

Since

$$\int u_j^2 u_1^2 < \frac{1}{2}\Big(\int u_j^4 + \int u_1^4\Big) < \int u_1^4,$$

we have

$$\int u_1^4 > \frac{1}{M-2}\sum_{j=2}^{M-1}\int u_j^2 u_1^2.$$

Finally, for (3.3.15) to hold, it is sufficient that

$$\sum_{j=2}^{M-1}\Big(b_{jM} - b_{1j} + \frac{b_{1M}-\mu_1}{M-2}\Big)\int u_j^2 u_1^2 > 0,$$

which is true under the conditions stated in Theorem 3.3.6. $\square$

3.3.2 Nonexistence of fully nontrivial ground states

Theorem 3.3.7 (Nonexistence Result I). *Let $M \geqslant 3$, $0 < \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_M$ and $b_{jm} \equiv b > 0$. There exists a constant $\alpha = \alpha(\omega_1, \omega_2)$ such that, if $(\omega_2, \ldots, \omega_M) \notin \mathcal{A}_\alpha$ and $b > \max\{\mu_1, \ldots, \mu_M\}$, then every ground state solution $\mathbf{u}$ of (3.3.1) is such that $u_m \equiv \ldots u_M \equiv 0$, for all m such that $\omega_m > \alpha\omega_2$. In particular, under these conditions, $G^* = \varnothing$.*

Proof. Along this proof, we will denote by $\mathbf{u}_b = (u_{1,b}, \ldots, u_{M,b})$ a solution of the system (3.3.1), highlighting the dependence on b. Since the proof is long, we divide it in three steps.

Step 1. Define $U_{m,b}(x) := \sqrt{b}\, u_{m,b}(x)$, which is a ground-state of

$$-\Delta U_{m,b} + \omega_m U_{m,b} = \frac{\mu_m}{b} U_{m,b}^3 + U_{m,b}\sum_{j\neq m} U_{j,b}^2. \tag{3.3.16}$$

We claim that there exists $C = C(\omega_1, \omega_2)$ such that $|U_{m,b}|^2_{\omega_m} \leqslant C$. Associated to system (3.3.16), we denote the corresponding action function by $I_{\omega,\frac{\mu}{b}}$, the Nehari manifold by $\mathcal{N}_{\omega,\frac{\mu}{b}}$, and the ground state level by $\mathcal{I}(\omega, \frac{\mu}{b})$. Observe that

$$\sum_{m=1}^{M} |U_{m,b}|^2_{\omega_m} = \mathcal{I}\left(\omega, \frac{\mu}{b}\right) \leqslant \mathcal{I}(\omega, 0)$$

by Lemma 3.1.3. On the other hand,

$$\begin{aligned}\mathcal{I}(\omega,0) &= \inf\{I_{\omega,0}(u) :\ u \neq 0,\ u \in \mathcal{N}_{\omega,0}\}\\ &\leqslant \inf\{I_{\omega,0}(u_1,u_2,0,\ldots,0) :\ (u_1,u_2)\neq(0,0),\ (u_1,u_2,0,\ldots,0)\in\mathcal{N}_{\omega,0}\}.\end{aligned}$$

Then $|U_{m,b}|^2_{\omega_m} \leqslant \mathcal{I}$, where $\mathcal{I}$ is the ground state level of the two equation system

$$\begin{cases} -\Delta w_1 + \omega_1 w_1 = w_1 w_2^2 \\ -\Delta w_2 + \omega_2 w_2 = w_2 w_1^2. \end{cases}$$

Thus the claim made above follows.

Step 2. Next, we prove that there exists $C = C(\omega_1, \omega_2)$ such that

$$\|\sqrt{b}\, u_{m,b}\|_\infty = \|U_{m,b}\|_\infty \leqslant C, \qquad \forall b > \max\{\mu_1, \dots, \mu_M\}. \tag{3.3.17}$$

Observe that, since $d \leqslant 3$, we have the continuous embedding $H^1(\mathbb{R}^d) \hookrightarrow L^6(\mathbb{R}^d)$.

Let us perform a standard Brézis-Kato type argument to pass from H^1 to L^∞ bounds. First of all, if there is a sequence $p_k \to \infty$ such that $\|U_{m,b}\|_{p_k} \leqslant 1$, the conclusion is obvious for $C := 1$. Suppose that $U_{m,b} \in L^{2+2\delta}(\mathbb{R}^d)$ for some $\delta > 0$. We test the equation for $U_{m,b}$ in (3.3.16) with $U_{m,b}^{1+\delta}$, obtaining

$$\frac{1+\delta}{(1+\delta/2)^2} \int \left(|\nabla U_{m,b}^{1+\delta/2}|^2 + \omega_m |U_{m,b}|^{2+\delta} \right) = \frac{\mu_m}{b} \int |U_{m,b}|^{4+\delta} + \int |U_{m,b}|^{2+\delta} \sum_{j \neq m} U_{j,b}^2.$$

Since $\omega_1 \leqslant \omega_m$ and $\mu_m/b \leqslant 1$, we deduce that

$$\begin{aligned} \min\left\{ \frac{1+\delta}{(1+\delta/2)^2}, \omega_1 \right\} \int \left(|\nabla |U_{m,b}|^{1+\delta/2}|^2 + \omega_m |U_m, b|^{2+\delta} \right) \\ \leqslant \int |U_{m,b}|^{1+\delta} (U_{m,b}^3 + U_{m,b} \sum_{j \neq m} U_{j,b}^2) \leqslant \|U_{m,b}\|_{2+2\delta}^{1+\delta} \|h_b\|_2 \end{aligned}$$

where $h_b := U_{m,b}^3 + U_{m,b} \sum_{j \neq m} U_{j,b}^2$. Thus

$$C_6^2 \min\left\{ \frac{1+\delta}{(1+\delta/2)^2}, \omega_1 \right\} \| U_{m,b}^{1+\delta/2} \|_6^2 \leqslant \|U_{m,b}\|_{2+2\delta}^{1+\delta} \|h_b\|_2$$

or, equivalently,

$$\|U_{m,b}\|_{6+3\delta} \leqslant \left(\frac{1}{C_6^2} \max\left\{ \frac{(1+\delta/2)^2}{1+\delta}, \frac{1}{\omega_1} \right\} \|h_b\|_2 \right)^{\frac{1}{2+\delta}} \|U_{m,b}\|_{2+2\delta}^{(1+\delta)/(2+\delta)}.$$

Assuming, without loss of generality, that $\|U_{m,b}\|_{2+2\delta} \geqslant 1$, then from the fact that $(1+\delta)/(2+\delta) \leqslant 1$ we deduce that $\|U_{m,b}\|_{2+2\delta}^{(1+\delta)/(2+\delta)} \leqslant \|U_{m,b}\|_{2+2\delta}$. Moreover, from the $H^1(\mathbb{R}^d)$ bound of Step 1, there exists $C = C(\omega_1, \omega_2)$ such that $\|h_b\|_2 \leqslant C$ for every $b > \max\{\mu_1, \dots, \mu_M\}$. Thus we conclude the existence of $\kappa = \kappa(\omega_1, \omega_2)$ such that

$$\|U_{m,b}\|_{6+3\delta} \leqslant \left(\kappa \max\left\{ \frac{(1+\delta/2)^2}{1+\delta}, \frac{1}{\omega_1} \right\} \right)^{\frac{1}{2+\delta}} \|U_{m,b}\|_{2+2\delta}.$$

Now we iterate, by letting

$$\delta(1) = 0, \qquad 2 + 2\delta(k+1) = 6 + 3\delta(k)$$

Observe that $\delta(k) \to \infty$, since $\delta(k) \geqslant (3/2)^{k-2}, k \geqslant 2$. With this choice of δ in the previous estimate, we obtain the iterative relation

$$\|U_{m,b}\|_{L^{2+2\delta(k+1)}} \leqslant \left(\kappa \max\left\{ \frac{(1+\delta/2)^2}{1+\delta}, \omega_m \right\} \right)^{\frac{1}{2+\delta}} \|U_{m,b}\|_{2+2\delta(k)}$$

which, together with $\delta(1) = 0$, gives

$$\begin{aligned}\|U_{m,b}\|_{3+6\delta(k)} &= \|U_{m,b}\|_{2+2\delta(k+1)} \leqslant \prod_{j=1}^{k} \left[\kappa \max\left\{\frac{(1+\delta/2)^2}{1+\delta}, \omega_m\right\}\right]^{\frac{1}{2+\delta(j)}} \|U_{m,b}\|_2 \\ &\leqslant \exp\left(\sum_{j=1}^{\infty} \frac{1}{2+\delta(j)} \log\left[\kappa \max\left\{\frac{(1+\delta(j)/2)^2}{1+\delta(j)}, \omega_m\right\}\right]\right) \|U_{m,b}\|_2\end{aligned}$$

As $\delta(j) \geqslant (3/2)^{j-2}, j \geqslant 2$, we see that

$$\sum_{j=1}^{\infty} \frac{1}{2+\delta(j)} \log\left[C\frac{(1+\delta(j)/2)^2}{1+\delta(j)}\right] < \infty,$$

which provides the uniform bound in $L^\infty(\mathbb{R}^d)$. Finally, it follows easily from scaling arguments that the bound C may be written as

$$C = C(\omega_1/\omega_2)\sqrt{\omega_2}. \tag{3.3.18}$$

Step 3. Take $i \geqslant 3$ such that $\omega_M \geqslant \ldots \omega_m > \alpha\omega_2 := MC^2$, where C is the constant appearing in (3.3.18). Take $j \in \{m, \ldots, M\}$. By multiplying the equation for $u_{j,b}$ by $u_{j,b}$ itself and recalling that $\mu_j/b \leqslant 1$ and (3.3.17) holds, we obtain

$$\begin{aligned}0 \leqslant \int |\nabla u_{j,b}|^2 &\leqslant \int \mu_j u_{j,b}^4 + \int b\, u_{j,b}^2 \sum_{k\neq j} u_{k,b}^2 - \int \omega_j u_{j,b}^2 \\ &\leqslant \int u_{j,b}^2(MC^2 - \omega_j) \leqslant 0,\end{aligned}$$

which implies that $u_{j,b} \equiv 0$. $\square$

Theorem 3.3.8 (Nonexistence Result II)**.** *Take a subset $P \subsetneq \{1, \ldots, M\}$ with $\#P \geqslant 2$. There exists a constant $B = B((\lambda_m, \mu_m)_{1\leqslant m\leqslant M}, (b_{jm})_{1\leqslant m,j\leqslant M,(j,m)\notin P^2})$ such that, if*

$$\min_{(j,m)\in P^2} b_{jm} > B,$$

then any ground state $\mathbf{u}$ of system (3.3.1) satisfies $u_{m_0} \equiv 0$ for every $m_0 \notin P$. In particular, $G^* = \varnothing$.

Proof. Let $(u_1, \ldots, u_M) \in G$. Define $\mathcal{I}((\omega_m; \mu_m; b_{jm})_{(j,m)\in P^2})$ to be the ground state level for the system

$$-\Delta w_m + \omega_m w_m = \mu_m w_m^3 + \sum_{j\in P} b_{jm} w_j^2 w_m,\ m \in P.$$

Then

$$I(u_1, \ldots, u_M) = \min_{(w_1,\ldots,w_M)\in\mathcal{N}_M} I(w_1, \ldots, w_M) \leqslant \min_{\substack{(w_1,\ldots,w_M)\in\mathcal{N}_M \\ w_m=0,\ i\notin P}} I(w_1, \ldots, w_M)$$

$$= \mathcal{I}((\omega_m; \mu_m; b_{jm})_{(j,m)\in P^2}).$$

Let $b = \min_{(j,m)\in P^2} b_{jm}$. From Lemma 3.1.3 and a simple normalization argument,

$$\mathcal{I}((\omega_m; \mu_m; b_{jm})_{(j,m)\in P^2}) \leqslant \mathcal{I}(\omega_m; 0, \dots, 0; b, \dots, b) = b^{-1}\mathcal{I}(\omega_m; 0, \dots, 0; 1, \dots, 1).$$

Hence

$$I(u_1, \dots, u_M) \leqslant b^{-1}\mathcal{I}(\omega_m; 0, \dots, 0; 1, \dots, 1).$$

Fix $i \notin P$ and suppose that $u_m \neq 0$. Then, for a constant $C > 0$, independent of b_{jm},

$$|u_m|^2_{\omega_m} = \mu_m \|u_m\|_4^4 + \int u_m^2 \sum_{j\neq m} b_{jm} u_j^2 \leqslant C\left(\mu_m |u_m|^4_{\omega_m} + |u_m|^2_{\omega_m} \sum_{j\neq m} b_{jm} |u_j|^2_{\omega_j}\right).$$

Therefore

$$\begin{aligned} 1 &\leqslant C\left(\mu_m |u_m|^2_{\omega_m} + \sum_{j\neq m} b_{jm}|u_j|^2_{\omega_j}\right) \leqslant 4C\left(\mu_m + \sum_{j\neq m} b_{jm}\right) I(u_1, \dots, u_M) \\ &\leqslant 4C\left(\mu_m + \sum_{j\neq m} b_{jm}\right) b^{-1}\mathcal{I}(\omega_m; 0, \dots, 0; 1, \dots, 1), \end{aligned}$$

which is absurd for b sufficiently large. □

Theorem 3.3.9 (Nonexistence Result III). *Let $b_{jm} \equiv b > 0$. Assume, without loss of generality, that $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_M$.*
Then, if

$$b < 2^{1-\frac{M}{2}}\sqrt{\mu_1\mu_M},$$

$G^* = \varnothing$.

Proof. Consider a fully nontrivial ground state $(u_1, u_2, \dots, u_M) \in \mathcal{N}_M$
We then compute $t > 0$ such that $(tu_1, 0, \dots, 0) \in \mathcal{N}_M$:

$$t^2|u_1|^2_{\omega_1} = t^4\mu_1\|u_1\|_4^4 \Leftrightarrow t^2 = \frac{|u_1|^2_{\omega_1}}{\mu_1\|u_1\|_4^4}.$$

Since $(u_1, u_2, \dots, u_M)$ is a ground state, $I(u_1, u_2, \dots, u_M) \leqslant I(tu_1, 0, \dots, 0)$, that is

$$\frac{1}{4}\Big(\sum_{m=1}^{M} \mu_m\|u_m\|_4^4 + 2b\sum_{m<j} \|u_m u_j\|_2^2\Big) \leqslant \frac{1}{4}\mu_1 t^4 \|u_1\|_4^4,$$

i.e.,

$$\mu_1\|u_1\|_4^4\Big(\sum_{m=1}^{M} \mu_m\|u_m\|_4^4 + 2b\sum_{m<j} \|u_m u_j\|_2^2\Big) \leqslant |u_1|^4_{\omega_1}.$$

Multiplying the first line of system (3.3.1) by u_1 and integrating then yields

$$\mu_1\|u_1\|_4^4\Big(\sum_{m=1}^{M}\mu_m\|u_m\|_4^4+2b\sum_{m<j}\|u_mu_j\|_2^2\Big)\leqslant\Big(\mu_1\|u_1\|_4^4+b\sum_{j=2}^{M}\|u_1u_j\|_2^2\Big)^2.$$

Hence

$$\begin{aligned}\mu_1\|u_1\|_4^4\sum_{m=2}^{M}\mu_m\|u_m\|_4^4+2b\mu_1\|u_1\|_4^4\sum_{m<j}\|u_mu_j\|_2^2\leqslant 2b\mu_1\|u_1\|_4^4\sum_{j=2}^{M}\|u_1u_j\|_2^2\\+b^2\Big(\sum_{j=2}^{M}\|u_1u_j\|_2^2\Big)^2,\end{aligned}$$

that is

$$\begin{aligned}\mu_1\|u_1\|_4^4\sum_{m=2}^{M}\mu_m\|u_m\|_4^4+2b\mu_1\|u_1\|_4^4\sum_{1<i<j}\|u_mu_j\|_2^2&\leqslant b^2\Big(\sum_{j=2}^{M}\|u_1u_j\|_2^2\Big)^2\\&\leqslant b^2\Big(\sum_{j=2}^{M}\|u_1\|_4^2\|u_j\|_4^2\Big)^2.\end{aligned}$$

Finally,

$$\sum_{m=2}^{M}\mu_1\mu_m\|u_m\|_4^4\leqslant b^2\Big(\sum_{j=2}^{M}\|u_j\|_4^2\Big)^2\leqslant b^22^{M-2}\sum_{j=2}^{M}\|u_j\|_4^4.$$

From this inequality, we obtain that

$$b^22^{M-2}\geqslant\mu_1\frac{\sum_{m=2}^{M}\mu_m\|u_m\|_4^4}{\sum_{m=2}^{M}\|u_m\|_4^4}\geqslant\mu_1\min\{\mu_m\,:\,m\neq 1\}.$$

By interchanging the roles of μ_1 and μ_j, $j\geqslant 2$, we get that for all $j\in\{1,\dots,M\}$,

$$b^22^{M-2}\geqslant\mu_j\min\{\mu_m\,:\,m\neq j\}.$$

In particular,

$$b\geqslant\frac{\sqrt{\mu_1\mu_M}}{2^{\frac{M}{2}-1}}.$$

□

Chapter 4

Dynamical implications

In this chapter, we study some properties of the initial value problem

$$\begin{cases} i(v_m)_t + \Delta v_m + \sum_{j=1}^{M} k_{jm}|v_j|^{p+1}|v_m|^{p-1}v_m = 0, \\ & m = 1, ..., M. \\ v_m(0) = (v_m)_0 \in H^1(\mathbb{R}^d) \end{cases} \qquad \text{(M-NLS}_t\text{)}$$

On the first section, we look at the global influence that ground-states have on the dynamics of (M-NLS$_t$). Afterwards, we study the stability of ground-states (and also some bound-states) as periodic solutions of (M-NLS$_t$). We recall (cf. Proposition 1.1.1) that the initial value problem is locally well-posed for $1 \leqslant p < 2/(d-2)^+$ and that, if the maximal time of existence $T(\mathbf{v}_0)$ is finite, then

$$\lim_{t \to T(\mathbf{v}_0)} \|\nabla \mathbf{v}(t)\|_2 = \infty,$$

4.1 Qualitative results

We recall that we are always assuming (PC). Moreover, we have

$$I(\mathbf{u}) = \sum_{m=1}^{M} \omega_m \|u_m\|_2^2 + \|\nabla u_m\|_2^2, \quad J(\mathbf{u}) = \sum_{j,m=1}^{M} k_{jm} \|u_j u_m\|_{p+1}^{p+1}.$$

Define

$$D = \{\mathbf{u} \in (H^1(\mathbb{R}^d))^M : J(\mathbf{u}) > 0\} \neq \varnothing$$

and, for each $\mathbf{u} = (u_1, \dots, u_M) \in (H^1(\mathbb{R}^d))^M$,

$$M(\mathbf{u}) = \sum_{m=1}^{M} \omega_m \|u_m\|_2^2, \quad K(\mathbf{u}) = \sum_{m=1}^{M} \|\nabla u_m\|_2^2, \quad E(\mathbf{u}) = \frac{1}{2}K(\mathbf{u}) - \frac{1}{2p+2}J(\mathbf{u})$$

$$GN(\mathbf{u}) = \frac{M(\mathbf{u})^{p+1-\frac{dp}{2}} K(\mathbf{u})^{\frac{dp}{2}}}{J(\mathbf{u})}.$$

Proposition 4.1.1. *The set of solutions for the minimization problem*

$$GN(\mathbf{u}) = \min_{\mathbf{w} \in D} GN(\mathbf{w}), \quad \mathbf{u} \in D \tag{4.1.1}$$

is G, up to scalar multiplication and scaling.

Proof. By Lemma 2.1.3 and Theorem 2.1.1, we know that $G \neq \varnothing$ is the set of solutions of

$$I(\mathbf{u}) = \min_{J(\mathbf{w})=\lambda_G} I(\mathbf{w}), \quad J(\mathbf{u}) = \lambda_G.$$

Let $\mathbf{u} \in G$ and $\mathbf{w} \in D$. Since $I(\mathbf{u}) = J(\mathbf{u}) > 0$, we have $\mathbf{u} \in D$. Define

$$\nu = \left(\frac{J(\mathbf{u})M(\mathbf{w})}{M(\mathbf{u})J(\mathbf{w})} \right)^{\frac{1}{2p}}$$

and

$$\zeta = \left(\nu^2 \left(\frac{M(\mathbf{w})}{M(\mathbf{u})} \right) \right)^{\frac{1}{d}}.$$

Then $\mathbf{z}(x) = \nu \mathbf{w}(\zeta x)$ satisfies

$$J(\mathbf{z}) = J(\mathbf{u}), \quad M(\mathbf{z}) = M(\mathbf{u}) \quad GN(\mathbf{z}) = GN(\mathbf{w}).$$

By the minimality of $\mathbf{u}$, $I(\mathbf{u}) \leqslant I(\mathbf{z})$, which implies that $GN(\mathbf{u}) \leqslant GN(\mathbf{z}) = GN(\mathbf{w})$. Therefore $\mathbf{u}$ is a solution of (4.1.1). On the other hand, if $\mathbf{w}$ is a solution of (4.1.1), then one has necessarily $GN(\mathbf{z}) = GN(\mathbf{u})$, which implies that $I(\mathbf{z}) = I(\mathbf{u})$. Therefore $\mathbf{z} \in G$, which concludes our proof. □

Set

$$C_M = GN(\mathbf{u})^{-1}, \ \mathbf{u} \in G.$$

Corollary 4.1.2. *The optimal constant $C > 0$ for the vector-valued Gagliardo-Nirenberg inequality*

$$\sum_{j,m=1}^{M} k_{jm} \|u_m u_j\|_{p+1}^{p+1} \leqslant C \left(\sum_{m=1}^{M} \omega_m \|u_m\|_2^2 \right)^{p+1-\frac{dp}{2}} \left(\sum_{m=1}^{M} \|\nabla u_m\|_2^2 \right)^{\frac{dp}{2}}, \ \mathbf{u} \in (H^1(\mathbb{R}^d))^M$$

is C_M.

Remark 4.1.1. In the special case $\omega_m \equiv 1$, one may compute more explicitly C_M: indeed, given $\mathbf{u} \in G$ with real nonnegative components, Theorem 2.2.4 implies that, up to rotations, $\mathbf{u} = (f_{max})^{-1/2p} Q\mathbf{y}$, where Q is the scalar ground-state and $\mathbf{y} \in (\mathbb{R}_0^+)^M$ satisfies

$$|\mathbf{y}| = 1, \quad \sum_{j,m=1}^{M} k_{jm} y_m^{p+1} y_j^{p+1} = f_{max} := \max_{|\mathbf{z}|=1} \sum_{j,m=1}^{M} k_{jm} |z_m|^{p+1} |z_j|^{p+1}.$$

Therefore

$$M(\mathbf{u}) = (f_{max})^{-1/p} M(Q\mathbf{y}) = (f_{max})^{-1/p} \|Q\|_2^2,$$
$$K(\mathbf{u}) = (f_{max})^{-1/p} K(Q\mathbf{y}) = (f_{max})^{-1/p} \|\nabla Q\|_2^2$$

and

$$J(\mathbf{u}) = (f_{max})^{-1-1/p} \left(\sum_{j,m=1}^{M} k_{jm} y_m^{p+1} y_j^{p+1} \right) \int |Q|^{2p+2} = (f_{max})^{-1/p} \int |Q|^{2p+2}$$

Hence the value of C_M is given by

$$C_M = f_{max} \frac{\|Q\|_{2p+2}^{2p+2}}{\|Q\|_2^{2p+2-dp} \|\nabla Q\|_2^{dp}} = f_{max} C_1$$

where C_1 is the optimal constant for the (scalar) Gagliardo-Nirenberg inequality (see [75]).

We now focus on the critical case $p = 2/d$.

REMARK 4.1.2. Let $\mathbf{w} \in A$. Since $p = 2/d$, the Pohozaev identity (see Proposition 1.2.4) implies that

$$E(\mathbf{w}) = 0.$$

Therefore

$$GN(\mathbf{u}) = \frac{M(\mathbf{u})^{\frac{2}{d}}}{p+1}, \quad \forall \mathbf{u} \in A.$$

Proposition 4.1.1 then implies that all ground-states have the same mass.

REMARK 4.1.3. It is easy to see that, for any $\mathbf{v} \in (H^1(\mathbb{R}^d))^M$,

$$\left(1 - \frac{1}{p+1} C_M M(\mathbf{v})^{\frac{2}{d}}\right) K(\mathbf{v}) \leqslant 2E(\mathbf{v}).$$

From the vector-valued Gagliardo-Nirenberg inequality, we have the following global existence result for (M-NLS$_t$):

Proposition 4.1.3. *Suppose that* $\mathbf{v}_0 \in (H^1(\mathbb{R}^d))^M$ *is such that*

$$M(\mathbf{v}_0) < \left(\frac{p+1}{C_M}\right)^{\frac{d}{2}} = M(\mathbf{u}),$$

with $\mathbf{u} \in G$. *Then* $T(\mathbf{v}_0) = \infty$.

Proof. It is a well-known fact that the functionals M and E are preserved by the flow generated by (M-NLS$_t$). Hence, if $\mathbf{v}$ is the solution of (M-NLS$_t$) with initial data $\mathbf{v}_0$, we have, by Remark 4.1.3, an uniform bound on $K(\mathbf{v}(\mathbf{t}))$ and so $T(\mathbf{v}_0) = \infty$. □

Remark 4.1.4. It is important to notice that Proposition 4.1.3 is valid for any vector $(\omega_1, \ldots, \omega_M) \in (\mathbb{R}^+)^M$. In fact, since system (M-NLS$_t$) preserves *individual* masses, there is an entire family of "total mass" norms which can be used in the qualitative results of this section.

The following result is an adaptation of the result in [41] to the vector case.

Lemma 4.1.4. *Suppose that* $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset (H^1(\mathbb{R}^d))^M$ *verifies*

1. $M(\mathbf{u}_n) = \overline{M} > 0\ \forall n \in \mathbb{N}$;
2. $K(\mathbf{u}_n) = \overline{K} > 0\ \forall n \in \mathbb{N}$;
3. $E(\mathbf{u}_n) \to 0$.

Then, given $\delta_0 > 0$, *up to a subsequence, there exist* $y_n \in \mathbb{R}^d$ *and* $R > 0$ *such that*

$$\sum_{m=1}^{M} \omega_m \int_{y_n + B_R} |(\mathbf{u}_n)_m|^2 dx \geqslant M(\mathbf{u}) - \delta_0,$$

where $\mathbf{u} \in G$. *If* $\overline{M} = M(\mathbf{u})$ *and* $\overline{K} = K(\mathbf{u})$, *then* $\mathbf{u}_n(\cdot - y_n) \to \mathbf{u}_0$ *in* $(H^1(\mathbb{R}^d))^M$ *and* $\mathbf{u}_0 \in G$.

Sketch of the proof: For a fixed $\epsilon > 0$, we apply Lemma 2.1.4 to the sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$. If the dichotomy case is true, $\mathbf{u}_n$ is decomposed in a localized part $\mathbf{u}_n^1$ and a remainder $\mathbf{u}_n^2$. Since $|M(\mathbf{u}_n^2) - (\overline{M} - \alpha)| < \epsilon$, up to a subsequence,

$$\lim M(\mathbf{u}_n^2) = \overline{M}_2 > 0,$$

Then one may apply again Lemma 2.1.4 to the sequence $\{\mathbf{u}_n^2\}_{n\in\mathbb{N}}$. This process may be iterated, thus decomposing $\mathbf{u_n}$ in a series of localized parts $\mathbf{u}_n^1, \ldots, \mathbf{u}_n^L$ plus a remainder $\mathbf{w}_n$ satisfying

$$\left| M(\mathbf{u}_n) - \sum_{l=1}^{L} M(\mathbf{u}_n^l) \right| < \epsilon, \quad K(\mathbf{u}_n) \geqslant \sum_{l=1}^{L} K(\mathbf{u}_n^l) - \epsilon,$$

and

$$\left| J(\mathbf{u}_n) - \sum_{l=1}^{L} J(\mathbf{u}_n^l) \right| < \epsilon.$$

By contradiction, suppose that, for any $\mathbf{u} \in G$,

$$M(\mathbf{u}_n^l) < M(\mathbf{u}) - \delta_0 = \left(\frac{p+1}{C_M} \right)^{\frac{d}{2}} - \delta_0, \quad l \leqslant L$$

Then, by the vector-valued Gagliardo-Nirenberg inequality (cf. Corollary 4.1.2) and by remark 4.1.3,

$$J(\mathbf{u}_n) \leqslant \sum_{l=1}^{L} J(\mathbf{u}_n^l) + \epsilon \lesssim \sum_{l=1}^{L} K(\mathbf{u}_n^l) + \epsilon$$

$$\lesssim \sum_{l=1}^{L} \left(1 - \frac{1}{p+1} C_M M(\mathbf{u}_n^l)^{\frac{2}{d}}\right)^{-1} E(\mathbf{u}_n^l) + \epsilon \lesssim E(\mathbf{u}_n) + \epsilon.$$

However, $E(\mathbf{u}_n) \to 0$ and $J(\mathbf{u}_n) \to (p+1)\overline{K}$, which is absurd. This proves the first part of the result.

If $\overline{M} = M(\mathbf{u})$, then the above argument shows that only the compactness alternative is possible (there can only exist one localized part). Since $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is bounded in $(H^1(\mathbb{R}^d))^M$, there exists $\mathbf{u}_0 \in (H^1(\mathbb{R}^d))^M$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$ and, from the compactness alternative, it follows that $\mathbf{u}_n \to \mathbf{u}_0$ in $(L^2(\mathbb{R}^d) \cap L^{2p+2}(\mathbb{R}^d))^M$ (see the proof of Theorem 2.1.1). In particular $M(\mathbf{u}_0) = M(\mathbf{u})$, $K(\mathbf{u}_0) \leqslant K(\mathbf{u})$ and

$$J(\mathbf{u}_0) = \lim J(\mathbf{u}_n) = (p+1)K(\mathbf{u}) = J(\mathbf{u}).$$

By the minimality of $\mathbf{u}$, we conclude that $\mathbf{u} \in G$. Moreover, $K(\mathbf{u}_0) = K(\mathbf{u}_0) = \lim K(\mathbf{u}_n)$, which implies the strong convergence $\mathbf{u}_n \to \mathbf{u}_0$ in $(H^1(\mathbb{R}^d))^M$. □

Using the previous lemma, one may prove the following results in the same way as in the scalar case $M = 1$:

Proposition 4.1.5 (L^2 concentration). *Fix $p = 2/d$. Let $\mathbf{v}_0 \in (H^1(\mathbb{R}^d))^M$ be such that $T = T(\mathbf{v}_0) < \infty$. Then, if $\mathbf{v}$ is the corresponding solution of* (M-NLS$_t$)*, there exists $x : [0, T(\mathbf{v}_0)) \to \mathbb{R}^d$ such that, for any $R > 0$,*

$$\liminf_{t \to T(\mathbf{v}_0)} \sum_{m=1}^{M} \omega_m \int_{|x - x(t)| < R} |\mathbf{v}_m(t)|^2 \geqslant M(\mathbf{u}),\ \mathbf{u} \in G.$$

Proposition 4.1.6 (Blowup profile). *Fix $p = 2/d$. Let $\mathbf{v}_0 \in (H^1(\mathbb{R}^d))^M$ be such that $T(\mathbf{v}_0) < \infty$ and $M(\mathbf{v}_0) = M(\mathbf{u})$, where $\mathbf{u} \in G$. Let $\mathbf{v}$ be the corresponding solution of* (M-NLS$_t$)*. Then, for any sequence $t_n \to T(\mathbf{v}_0)$, there exists $\mathbf{u}_0 \in G$ and $y_n \in \mathbb{R}^d$ such that*

$$\left(\frac{K(\mathbf{u}_0)}{K(\mathbf{v}(t_n))}\right)^{\frac{N}{4}} \mathbf{v}\left(\left(\frac{K(\mathbf{u}_0)}{K(\mathbf{v}(t_n))}\right)^{\frac{1}{2}} \cdot + y_n, t_n\right) \to \mathbf{u}_0 \textit{ in } (H^1(\mathbb{R}^d))^M$$

4.2 Stability of bound-states

In this section, we study the stability of bound-states with respect to the flow generated by (M-NLS$_t$). The scalar case is treated in [12], [65] and [38], among others. For ground-states, stability is equivalent to the condition $p < 2/d$ (the L^2-subcritical case - see Appendix B). Notice that, from the gauge and translation invariances, one should study the orbital stability of ground-states (that is, modulo rotations and translations). In [12], it is possible to find examples which show that one must really consider this kind of stability.

For the general case of bound-states, the problem is much more difficult. It can be seen that, assuming non-degeneracy, the orbital stability of a bound-state is directly related with the Morse index of the action at the bound-state (see [38]). If this index is 1, then stability is again equivalent to the condition $p < 2/d$. If the index is greater than 1, the problem remains open. We observe that the assumption of non-degeneracy is not always true, even for ground-states: in the hypothesis of Corollary 2.3.2, there exists a continuum of ground-states which are not related by gauge invariance. This situation, though somewhat excepcional, shows that one cannot use *a priori* the results of [38].

Here, we show the analogous stability results for ground-states of (M-NLS$_t$), assuming only the positivity condition (PC). This was done for $M = 2$ and $k_{jm} > 0$ in [55]. The framework will be very close to the scalar case as is [12], though some subtle changes will be done. Specifically, to prove stability (or instability), one proves that the set of ground-states is the set of minimizers of an adequate minimization problem. This is done in two steps:

1. Prove that the minimization problem has a solution, independently of the existence of ground-states;
2. Using the solution found in the previous step, show the equivalence between ground-states and minimizers.

Here, we change the argument. We shall prove directly that ground-states are minimizers and conclude the equivalence. This is more efficient, since the proof of existence of minimizers without using the ground-states and assuming only (PC) has to go through the concentration-compactness principle. Furthermore, we define three different classes of bound-states and prove stability results for these solutions. These are generalizations of the results obtained in [55].

Define, for any $\mathbf{u} \in (H^1(\mathbb{R}^d))^M$,

$$H(\mathbf{u}) := K(\mathbf{u}) - \frac{dp}{2p+2} J(\mathbf{u}), \quad S(\mathbf{u}) = \frac{1}{2} I(\mathbf{u}) - \frac{1}{2p+2} J(\mathbf{u}).$$

If $\mathbf{u} \in A$, Pohozaev's identity (cf. Proposition 1.2.4) implies that $H(\mathbf{u}) = 0$.

Remark 4.2.1. Consider, for $\mathbf{u} \in (H^1(\mathbb{R}^d))^M$ and $\lambda > 0$, $\mathcal{P}(\mathbf{u}, \lambda)(x) = \lambda^{\frac{d}{2}} \mathbf{u}(\lambda x)$. By a change of variables, one sees that

$$M(\mathcal{P}(\mathbf{u}, \lambda)) = M(\mathbf{u}).$$

Now, differentiating $S(\mathcal{P}(\mathbf{u}, \lambda))$ with respect to λ,

$$\frac{d}{d\lambda} S(\mathcal{P}(\mathbf{u}, \lambda)) = H(\mathcal{P}(\mathbf{u}, \lambda)). \tag{4.2.1}$$

Remark 4.2.2. As in the scalar case, one may prove the Virial identity for (M-NLS$_t$) (see Appendix C): given $\mathbf{v} = (v_1, ..., v_M) : [0, T) \to (H^1(\mathbb{R}^d))^M$ solution of (M-NLS$_t$), one has

$$\frac{d^2}{dt^2} \sum_{m=1}^{M} \|x v_m(t)\|_2^2 = 8H(\mathbf{v}(t)). \tag{4.2.2}$$

The quantity $\sum_{m=1}^{M} \|xv_m(t)\|_2^2$ is called the variance of $\mathbf{v}(t)$. This identity will be essential to prove instability.

Definition 4.2.1. *Let $\mathcal{S} \subset (H^1(\mathbb{R}^d))^M$ be invariant by the flow generated by (M-NLS$_t$). We say that $\mathcal{S}$ is:*

1. *stable if, for each $\delta > 0$, there exists an $\epsilon > 0$ such that, for any $\mathbf{v}_0 \in (H^1(\mathbb{R}^d))^M$ with*
$$\inf_{\mathbf{w}\in\mathcal{S}} \|\mathbf{v}_0 - \mathbf{w}\|_{(H^1(\mathbb{R}^d))^M} < \epsilon,$$
the solution $\mathbf{v}$ of (M-NLS$_t$) with initial data $\mathbf{v}_0$ satisfies
$$\inf_{\mathbf{w}\in\mathcal{S}} \|\mathbf{v}(t) - \mathbf{w}\|_{(H^1(\mathbb{R}^d))^M} < \delta, \forall t < T(\mathbf{v}_0).$$

2. *weakly unstable if there exist $\epsilon > 0$ and a sequence $\mathbf{v}_0^n$ such that*
$$\inf_{\mathbf{w}\in\mathcal{S}} \|\mathbf{v}_0^n - \mathbf{w}\|_{(H^1(\mathbb{R}^d))^M} \to 0, \ n \to \infty$$
and, letting $\mathbf{v}^n$ be the solution of (M-NLS$_t$) with initial data $\mathbf{v}_0^n$,
$$\sup_{t\in[0,T(\mathbf{v}_0^n))} \inf_{\mathbf{w}\in\mathcal{S}} \|\mathbf{v}^n(t) - \mathbf{w}\|_{(H^1(\mathbb{R}^d))^M} > \epsilon.$$

3. *unstable if, for any $\mathbf{v}_0 \in \mathcal{S}$, there exists a sequence $\mathbf{v}_0^n \to \mathbf{v}_0$ such that $T(\mathbf{v}_0^n) < \infty$, for any $n \in \mathbb{N}$.*

Lemma 4.2.2. *There exists $\mu > 0$ such that*
$$M(\mathbf{u}) = \mu, \ \forall \mathbf{u} \in G.$$

Proof. This follows easily from the identities $I(\mathbf{u}) = J(\mathbf{u})$ and Pohozaev's identity. □

Lemma 4.2.3. *Fix $p < 2/d$ and assume (PC). Then G is the set of of solutions of the minimization problem*
$$E(\mathbf{u}) = \min_{M(\mathbf{w})=\mu} E(\mathbf{w}), \quad M(\mathbf{u}) = \mu. \tag{4.2.3}$$
Moreover, if $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, then $J(\mathbf{w}_n) \to J(\mathbf{u})$, with $\mathbf{u} \in G$.

Proof. Let $\mathbf{w}$ be such that $M(\mathbf{w}) = \mu$. Consider the function (see Remark 4.2.1)
$$\lambda \mapsto f(\lambda) = E(\mathcal{P}(\mathbf{w}, \lambda)) = S(\mathcal{P}(\mathbf{w}, \lambda)) - \frac{1}{2}M(\mathbf{w}), \ \lambda > 0$$
Since $p < 2/d$, f has a unique minimizer λ_0. Let $\mathbf{z} = \mathcal{P}(\mathbf{w}, \lambda_0)$. Then $f'(\lambda_0) = 0$, which implies that $H(\mathbf{z}) = 0$, i.e.,
$$K(\mathbf{z}) = \frac{dp}{2p+2} J(\mathbf{z}).$$

Therefore,

$$E(\mathbf{z}) = \frac{dp-2}{2dp} K(\mathbf{z}).$$

Using the vector-valued Gagliardo-Nirenberg inequality,

$$\frac{2p+2}{dp} K(\mathbf{z}) = J(\mathbf{z}) \leqslant C_M M(\mathbf{z})^{\frac{2-(d-2)p}{2}} K(\mathbf{z})^{\frac{dp}{2}}, \tag{4.2.4}$$

and so, from $M(\mathbf{z}) = \mu$,

$$\frac{2p+2}{dp} K(\mathbf{z})^{\frac{2-dp}{2}} \leqslant C_M \mu^{\frac{2-(d-2)p}{2}}.$$

Let $\mathbf{u} \in G$. By Lemma 4.1.1, we obtain $K(\mathbf{z}) \leqslant K(\mathbf{u})$. Therefore

$$E(\mathbf{w}) \geqslant E(\mathbf{z}) = \frac{dp-2}{2Np} K(\mathbf{z}) \geqslant \frac{dp-2}{2dp} K(\mathbf{u}) = E(\mathbf{u})$$

and so $\mathbf{u}$ is a solution of (4.2.3). If $\mathbf{w}$ is also a solution of (4.2.3), then one must have equality in (4.2.4). Again by Lemma 4.1.1,

$$\nu \mathbf{w}(\zeta x) \in G,$$

with

$$\nu = \left(\frac{J(\mathbf{u}) M(\mathbf{w})}{M(\mathbf{u}) J(\mathbf{w})} \right)^{\frac{1}{2p}}$$

and

$$\zeta = \left(\nu^2 \left(\frac{M(\mathbf{w})}{M(\mathbf{u})} \right) \right)^{\frac{1}{d}}.$$

Since $M(\mathbf{w}) = M(\mathbf{u})$ and $J(\mathbf{w}) = J(\mathbf{u})$, $\nu = \zeta = 1$. Therefore $\mathbf{w} \in G$.

If $\{\mathbf{w}_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, define $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ as above. Then $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ is also a minimizing sequence and

$$\|\mathbf{w}_n - \mathbf{z}_n\|_{(H^1(\mathbb{R}^d))^M} \to 0, \ n \to \infty.$$

Hence

$$\frac{dp-2}{2(2p+2)} J(\mathbf{u}) = E(\mathbf{u}) = \lim E(\mathbf{z}_n) = \lim \frac{dp-2}{2(2p+2)} J(\mathbf{z}_n) = \lim \frac{dp-2}{2(2p+2)} J(\mathbf{w}_n),$$

as we wanted. □

Definition 4.2.4. *Fix $X \subset \{1, \dots, M\}$ and let $L = |X|$. An element $U \in A$ belongs to G_X if the vector of its nonzero components is a ground-state for the (L-NLS) system formed by the components $j \in X$.*

Theorem 4.2.5. *Assume (PC) and $p < 2/d$. For any $X \subset \{1, ..., M\}$, let $G_1 \subset G_X$ be such that dist$(G_1, G_X \backslash G_1) > \delta$, for some $\delta > 0$. Then G_1 is stable.*

Proof. We start with the stability for $X = \{1, ..., M\}$ (that is, for G). By contradiction suppose that there exists a sequence $\{\mathbf{v}_0^n\}_{n\in\mathbb{N}} \subset (H^1(\mathbb{R}^d))^M$ such that, for some $\mathbf{u}_0 \in G_1$,

$$\|\mathbf{v}_0^n - \mathbf{u}_0\|_{(H^1(\mathbb{R}^d))^M} \to 0,\ n \to \infty$$

and, letting $\mathbf{v}_n$ be the solution of (M-NLS$_t$) with initial data $\mathbf{v}_0^n$, there exist $\{t_n\}_{n\in\mathbb{N}}$ and $\epsilon > 0$ such that

$$\inf_{\mathbf{u}\in G_1} \|\mathbf{v}^n(t_n) - \mathbf{u}\|_{(H^1(\mathbb{R}^d))^M} = \epsilon.$$

By continuity and conservation of mass and energy,

$$E(\mathbf{v}^n(t_n)) = E(\mathbf{v}_0^n) \to E(\mathbf{u}_0),\ M(\mathbf{v}^n(t_n)) = M(\mathbf{v}_0^n) \to M(\mathbf{u}_0) = \mu.$$

Therefore, the sequence

$$\mathbf{w}_n = \left(\frac{\mu}{M(\mathbf{v}_0^n)}\right)^{\frac{1}{2}} \mathbf{v}^n(t_n)$$

is a minimizing sequence of (4.2.3). By Lemma 4.2.3, $J(\mathbf{w}_n) \to J(\mathbf{u}_0)$. Since $I(\mathbf{w}_n) \to I(\mathbf{u}_0)$, it follows from Theorem 2.1.1 that $\mathbf{w}_n \to \mathbf{u}_1$, with $\mathbf{u}_1 \in G$, which implies that $\mathbf{v}^n(t_n) \to \mathbf{u}_1$. Taking $\epsilon < \delta$, one obtains $d(\mathbf{u}_1, G_1) < \delta$, which means that $\mathbf{u}_1 \in G_1$, which is absurd.

In the general case, given $X \subset \{1, ..., M\}$, one may proceed exactly as above: for $i \notin X$, since the mass of each component is conserved, the i-th components must converge to 0 in L^2 and, by interpolation, to 0 in L^{2p+2}. This means that the remaining components are a minimizing sequence of (4.2.3), for the (L-NLS) system formed by the components in X, and therefore must converge to a ground-state of such a system. ☐

Remark 4.2.3. In many cases, the set G is discrete modulo translations and rotations. Then any connected component of G is stable. Since these components are obtained by translations and rotations of a given element, one obtains orbital stability of ground-states.

In Lemma 4.2.3, one proves that, if $p < 2/d$, the set of ground-states may be obtained by minimizing the energy given a fixed mass. If instead one minimizes the energy given fixed *individual* masses,

$$E(\mathbf{u}) = \min_{\{\mathbf{w}:\|w_m\|_2^2 = c_m\}} E(\mathbf{w}), \quad \|\mathbf{u}_m\|_2^2 = c_m > 0, \tag{4.2.5}$$

one obtains bound-states with possibly incoherent components (the existence of solutions may be ensured by the concentration-compactness principle). Since these bound-states are always nontrivial, there are cases where they are not ground-states. However, one may still prove that these solutions form a stable set, just by adapting the proof of Theorem 4.2.5. In some cases, one may still obtain characterization results for (4.2.5):

Proposition 4.2.6. *Suppose (PC), $p < 2/d$, $k_{jm} > 0, m \neq j$ and that there exists $\beta > 0$ such that*

$$\sum_{j=1}^{M} k_{jm} = \beta, \ \forall m.$$

Then, for $c = \|\beta^{\frac{1}{2p}} Q\|_2^2$, the set B^c of solutions to (4.2.5) *with $c_m \equiv c$ is given by*

$$B^c = \{(e^{i\theta_m}\beta^{\frac{1}{2p}} Q(\cdot + y))_{1\leqslant m\leqslant M} : \theta_m \in \mathbb{R}, y \in \mathbb{R}^d\}.$$

Proof. Define, for $u \in H^1(\mathbb{R}^d)$,

$$E_1(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\beta}{2p+2}\|u\|_{2p+2}^{2p+2}.$$

By [12, Corollary 8.3.8], the set of solutions of the minimization problem

$$E_1(u) = \min_{\|w\|_2^2 = \|\beta^{\frac{1}{2p}} Q\|_2^2} E_1(w), \quad \|u\|_2^2 = \|\beta^{\frac{1}{2p}} Q\|_2^2$$

is $\{e^{i\theta}\beta^{\frac{1}{2p}} Q(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^d\}$. Let $\mathbf{u} = (u_1, ..., u_M) \in (H^1(\mathbb{R}^d))^M$ be such that $\|u_m\|_2^2 = \|\beta^{\frac{1}{2p}} Q\|_2^2$. Then

$$\sum_{m=1}^{M} E_1(u_m) \geqslant \sum_{m=1}^{M} E_1(\beta^{\frac{1}{2p}} Q).$$

Let $\mathcal{Q}$ be the vector formed by M copies of $\beta^{\frac{1}{2p}} Q$. Now, from Young's inequality, we have

$$E(\mathbf{u}) \geqslant \sum_{m=1}^{M} E_1(u_m) \geqslant \sum_{m=1}^{M} E_1(\beta^{\frac{1}{2p}} Q) = E(\mathcal{Q}).$$

Therefore $\mathcal{Q}$ is a solution of (4.2.5). If $\mathbf{u}$ is also a solution, one must have equality in the above relation, which implies that

$$u_m = e^{i\theta_m}\beta^{\frac{1}{2p}} Q(\cdot + y_m), \ \theta_m \in \mathbb{R}, \ y_m \in \mathbb{R}^d.$$

If there exist m_0, j_0 such that $y_{m_0} \neq y_{j_0}$, one easily sees that there exists $D \subset \mathbb{R}^d$ of positive measure such that, for all $x \in D$, $Q(x + y_{m_0}) \neq Q(x + y_{m_0})$. Using Young's inequality,

$$Q(x + y_{m_0})^{p+1} Q(x + y_{j_0})^{p+1} < \frac{1}{2} Q(x + y_{m_0})^{2p+2} + \frac{1}{2} Q(x + y_{j_0})^{2p+2}, \quad x \in D.$$

On the other hand, we have in general

$$Q(x+y_m)^{p+1} Q(x+y_j)^{p+1} \leqslant \frac{1}{2} Q(x+y_m)^{2p+2} + \frac{1}{2} Q(x+y_j)^{2p+2}, \quad x \in \mathbb{R}^d, \ 1 \leqslant m, j \leqslant M.$$

Consequently,

$$\begin{aligned}
&\int (a_m Q(\cdot + y_m))^{p+1}(a_j Q(\cdot + y_j))^{p+1} \\
&\leqslant a_m^{p+1}a_j^{p+1}\left(\frac{1}{2}\int Q(\cdot + y_m)^{2p+2} + \frac{1}{2}\int Q(\cdot + y_j)^{2p+2}\right) \\
&= a_m^{p+1}a_j^{p+1}\int Q^{2p+2} = \int (a_m Q)^{p+1}(a_j Q)^{p+1},
\end{aligned}$$

with strict inequality if $m = m_0$ and $j = j_0$. Thus

$$E(\mathbf{u}) = \frac{1}{2}K(\mathbf{u}) - \frac{1}{2p+2}J(\mathbf{u}) > \frac{1}{2}K(\mathcal{Q}) - \frac{1}{2p+2}J(\mathcal{Q}) = E(\mathcal{Q}),$$

which is absurd. Hence

$$B^c = \{(e^{i\theta_m}\beta^{\frac{1}{2p}}Q(\cdot + y))_{1\leqslant m\leqslant M} : \theta_m \in \mathbb{R}, y \in \mathbb{R}^d\}.$$

□

Now we turn our attention to the case $p \geqslant 2/d$. Define, for $\mathbf{w} \neq 0$, $\lambda^*(\mathbf{w})$ to be the maximum of the function $g(\lambda) = S(\mathcal{P}(\mathbf{w}, \lambda))$.

Lemma 4.2.7. *Assume (PC) and $p > 2/d$. Then G is the set of of solutions of the minimization problem*

$$S(\mathbf{u}) = \min_{H(\mathbf{w})=0} S(\mathbf{w}), \quad H(\mathbf{u}) = 0. \tag{4.2.6}$$

Proof. By (4.2.1), one may check that $\lambda^*(\mathbf{w}) = 1$. Therefore

$$S(\mathcal{P}(\lambda, \mathbf{w})) \leqslant S(\mathbf{w}), \forall \lambda > 0.$$

On the other hand, there exists $\lambda_0 > 0$ such that $J(\mathcal{P}(\lambda_0, \mathbf{u})) = \lambda_G$. Hence, for $\mathbf{u} \in G$,

$$S(\mathbf{u}) \leqslant S(\mathcal{P}(\lambda_0, \mathbf{w})) \leqslant S(\mathbf{w}), \forall \mathbf{w} : \ H(\mathbf{w}) = 0.$$

Therefore G is a subset of the set of solutions of (4.2.6) and the latter is nonempty.

Now consider $\mathbf{v}$ solution of (4.2.6). Define, for $\sigma > 0$, $\mathbf{v}_\sigma(x) = \sigma^{\frac{1}{p}}\mathbf{v}(\sigma x)$. By a change of variables,

$$H(\mathbf{v}_\sigma) = \sigma^{2-d+\frac{2}{p}}H(\mathbf{v}) = 0.$$

Since $\mathbf{v}$ is a minimizer, one must have

$$\frac{d}{d\sigma}S(\mathbf{v}_\sigma)\Big|_{\sigma=1} = 0, \text{ i.e. } \langle S'(\mathbf{v}), \mathbf{v}\rangle_{H^{-1}\times H^1} = 0$$

On the other hand, there exists η such that $S'(\mathbf{v}) = \eta H'(\mathbf{v})$. Applying to $\mathbf{v}$ and using $H(\mathbf{v}) = 0$,

$$0 = \langle S'(\mathbf{v}), \mathbf{v}\rangle_{H^{-1}\times H^1} = \eta\langle H'(\mathbf{v}), \mathbf{v}\rangle_{H^{-1}\times H^1} = -2p\eta K(\mathbf{v}).$$

Therefore $\eta = 0$ and so $\mathbf{v} \in A$. Given $\mathbf{u} \in G$, $H(\mathbf{u}) = 0$, and so $S(\mathbf{u}) \leqslant S(\mathbf{v})$, which means that $\mathbf{v} \in G$. □

Lemma 4.2.8. *Let $\mathbf{u} \in G$ and $\mathbf{w} \in (H^1(\mathbb{R}^d))^M$ such that $H(\mathbf{w}) < 0$. Then*

$$H(\mathbf{w}) \leqslant S(\mathbf{w}) - S(\mathbf{u}).$$

Proof. Since $H(\mathbf{w}) < 0$, one has $\lambda^*(\mathbf{w}) < 1$. Moreover, the mapping $\lambda \mapsto S(\mathcal{P}(\mathbf{w}, \lambda))$ is concave in $(\lambda^*(\mathbf{w}), 1)$. Therefore, by Remark 4.2.1,

$$S(\mathbf{w}) \geqslant S(\mathcal{P}(\mathbf{w}, \lambda^*(\mathbf{w}))) + (1-\lambda^*(\mathbf{w}))H(\mathbf{w}) \geqslant S(\mathcal{P}(\mathbf{w}, \lambda^*(\mathbf{w}))) + H(\mathbf{w}) \geqslant S(\mathbf{u}) + H(\mathbf{w}),$$

since $H(\mathcal{P}(\mathbf{w}, \lambda^*(\mathbf{w}))) = 0$ and $\mathbf{u}$ is a solution of (4.2.6). □

Remark 4.2.4. More generally, given any $\mathbf{v} \in (H^1(\mathbb{R}^d))^M$, one may prove as above that, if $\mathbf{w}$ is such that $H(\mathbf{w}) < 0$ and $S(\mathcal{P}(\mathbf{w}, \lambda_0)) \geqslant S(\mathbf{v})$,

$$H(\mathbf{w}) \leqslant S(\mathbf{w}) - S(\mathbf{v}).$$

Definition 4.2.9. *We denote by R the set of bound-states such that all nonzero components are equal to the same ground-state of (NLS), up to scalar multiplication and rotation.*

Theorem 4.2.10. *Assume (PC) and $p > 2/d$. Then G and R are unstable.*

Proof. Firstly, we prove that G is unstable. Consider a ground-state $\mathbf{u}$. Then, for any $\lambda > 1$, $\mathbf{u}_\lambda := \mathcal{P}(\mathbf{u}, \lambda)$ satisfies

$$H(\mathbf{u}_\lambda) < 0.$$

Let $\mathbf{v}_\lambda$ be the solution of (M-NLS) with initial data $\mathbf{u}_\lambda$. For t small, $H(\mathbf{v}_\lambda(t)) < 0$. From the conservation of mass and energy,

$$S(\mathbf{v}_\lambda(t)) = S(\mathbf{u}_\lambda).$$

By the previous Lemma, for any t such that $H(\mathbf{v}_\lambda(t)) < 0$, one has

$$H(\mathbf{v}_\lambda(t)) \leqslant S(\mathbf{v}_\lambda(t)) - S(\mathbf{u}) \leqslant S(\mathbf{u}_\lambda) - S(\mathbf{u}) = -\delta < 0.$$

Therefore, by continuity, one must have $H(\mathbf{v}_\lambda(t)) \leqslant -\delta, \forall t < T(\mathbf{u}_\lambda)$. Now, using (4.2.2),

$$\frac{d^2}{dt^2} \sum_{m=1}^{M} \|x(\mathbf{v}_\lambda)_m(t)\|_2^2 = 8H(\mathbf{v}_\lambda(t)) < -8\delta. \tag{4.2.7}$$

Since the variance is positive, one must have $T(\mathbf{v}_\lambda) < \infty$ and so G is unstable.

If $\mathbf{u} = (u_1, ..., u_M) \in R$, then there exist $a_m \geqslant 0$, $\theta_m \in \mathbb{R}$ and $y \in \mathbb{R}^d$ such that $u_m = a_m e^{i\theta_m} Q(\cdot + y)$. Since $Q(\cdot + y)$ is a ground-state for (1-NLS), there exists a sequence $\{v_n^0\}_{n\in\mathbb{N}}$ such that $v_n^0 \to Q(\cdot + y)$ in $H^1(\mathbb{R}^d)$ and $T(v_n^0) < \infty$, $\forall n$. Let v_n be the solution of (1-NLS) with initial data v_n^0. Then one can observe that $\mathbf{v}_n = (a_m e^{i\theta_m} v_n)_{1\leqslant m\leqslant M}$ is a solution of (M-NLS), with initial data $\mathbf{v}_n^0 = (a_m e^{i\theta_m} v_n^0)_{1\leqslant m\leqslant M}$. Since $\mathbf{v}_n^0 \to U$ in $(H^1(\mathbb{R}^d))^M$, one concludes that R is unstable. □

Theorem 4.2.11. *Assume (PC) and $p > 2/d$. If $\mathbf{u} \in A$ is a local minimum of S over the set*

$$\mathcal{H} := \{\mathbf{w} : \ H(\mathbf{w}) = 0\},$$

then the set $\{e^{i\theta}\mathbf{u}(\cdot + y) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^d\}$, is weakly unstable.

Proof. Let $\mathbf{u} \in A$ be a local minimum of S restricted to $\mathcal{H}$. Let $B_\delta(\mathbf{u})$ be a ball with center at $\mathbf{u}$ and radius δ fixed such that

$$S(\mathbf{u}) \leqslant S(\mathbf{w}), \ \forall \mathbf{w} \in B_\delta(\mathbf{u}) \cap \mathcal{H}.$$

For $\epsilon > 0$ small, one has

$$\mathcal{P}(\mathbf{w}, \lambda^*(\mathbf{w})) \in B_\delta(\mathbf{u}), \ \forall \mathbf{w} \in B_\epsilon(U).$$

From Remark 4.2.4, if $\mathbf{w} \in B_\epsilon(\mathbf{u})$ is such that $H(\mathbf{w}) < 0$,

$$H(\mathbf{w}) \leqslant S(\mathbf{w}) - S(\mathbf{u}).$$

Notice that, from the invariance of S and H regarding rotations and translations, the same remains valid for

$$\mathbf{w} \in \Sigma := \{e^{i\theta}\mathbf{z}(\cdot + y) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^d, \ \mathbf{z} \in B_\epsilon(\mathbf{u})\}.$$

Consider $\mathbf{u}_\lambda = \mathcal{P}(\mathbf{u}, \lambda)$, $\lambda > 1$. Then $H(\mathbf{u}_\lambda) < 0$. Let $\mathbf{v}_\lambda$ be the solution of (M-NLS) with initial data $\mathbf{u}_\lambda$. If $\mathbf{v}_\lambda(t) \in \Sigma$, $\forall t < T(\mathbf{u}_\lambda)$, then, arguing as in the previous proof,

$$H(\mathbf{v}_\lambda(t)) \leqslant S(\mathbf{u}_\lambda) - S(\mathbf{u}) = -\delta < 0, \ \forall t < T(\mathbf{u}_\lambda).$$

Then (4.2.7) is valid, which leads to $T(\mathbf{u}_\lambda) < \infty$. Since Σ is bounded, we arrive at a contradiction. ☐

Remark 4.2.5. Set $p > 2/d$. Assuming that $\mathbf{u} \in A$ is a non-degenerate critical point of the action (modulo rotations), the Morse index of S at $\mathbf{u}$, $m(\mathbf{u})$, is greater or equal to 1: a negative direction is given by the path $\lambda \mapsto S(\mathcal{P}(\mathbf{u}, \lambda))$. Notice that this direction does not belong to the tangent space of $\mathcal{H}$ at $\mathbf{u}$. Therefore, the condition in the above theorem is equivalent $m(\mathbf{u}) = 1$. The problem for $m(\mathbf{u}) \geqslant 2$ is much more difficult, and it is still unanswered for the scalar equation.

Theorem 4.2.12. *Assume (PC) and $p = 2/d$. Then A is unstable.*

Proof. First, notice that $2E(\mathbf{w}) = H(\mathbf{w})$, for any $\mathbf{w} \in (H^1(\mathbb{R}^d))^M$. Let $\mathbf{u} \in A$. Then $2E(\mathbf{u}) = H(\mathbf{u}) = 0$. For any $\lambda > 1$, $H(\lambda\mathbf{u}) < 0$. Since $2E = H$, the conservation of energy implies that, setting $\mathbf{v}_\lambda$ to be the solution of (M-NLS) with initial data $\lambda\mathbf{u}$, $H(\mathbf{v}_\lambda(t)) = H(\lambda\mathbf{u})$, $t < T(\lambda\mathbf{u})$. One now concludes as in the supercritical case, using the Virial identity. ☐

Part II

Dynamics for some variants of the nonlinear Schrödinger equation

Chapter 5

Some L^∞ solutions of the hyperbolic nonlinear Schrödinger equation

5.1 Introduction

In this chapter, we shall consider the hyperbolic nonlinear Schrödinger equation

$$iu_t + u_{xx} - \Delta_{\mathbf{y}} u + \lambda |u|^\sigma u = 0, \tag{HNLS}$$

where

$$\lambda \in \mathbb{R}, \quad 0 < \sigma < 4/(d-2)^+, \quad u = u(t, x, \mathbf{y}), (x, \mathbf{y}) \in \mathbb{R}^d,\ d \geqslant 2.$$

This is a special case of a larger class of dispersive equations, namely

$$iu_t - Lu + \lambda |u|^\sigma u = 0,$$

with L a second-order differential operator on the spatial variables. In fact, (HNLS) corresponds to the case where the principal part of L has a unique negative direction. It is important to notice that, when all directions are positive, L is elliptic, which includes the nonlinear Schrödinger equation.

Physically, (HNLS) is related with deep water waves and plasma physics (for $d = 2$) and nonlinear optics (for $d = 3$); see [68], [18]. In the latter case, t is to be interpreted as the propagation direction and x is time. Due to its practical relevance, this equation has been analyzed numerically in several papers (see [73], [68]). On the other hand, from a theoretical point of view, there exist works on both the linear equation and on some special solutions to (HNLS) (see [31], [30], [46], [53], [36]). However, a qualitative theory is still to be discovered. In fact, even though there is a very complete set of techniques for (NLS), the presence of a negative direction makes most of them unusable, as we shall see later on.

In this chapter, we shall build examples of solutions for the initial value problem associated with (HNLS),

$$\begin{cases} iu_t + \Box u + \lambda|u|^\sigma u = 0, & \Box := \partial^2_{xx} - \Delta_{\mathbf{y}} \\ u(0,x,\mathbf{y}) = u_0(x,\mathbf{y}) & . \end{cases} \tag{5.1.1}$$

Interestingly enough, even though these solutions present all sorts of qualitative behaviour (global existence, blow-up, etc.), none of them will be in $H^1(\mathbb{R}^d)$.

In Section 5.2, we present a large family of L^∞ (semiclassical) solutions, which present both global and blow-up behaviour (in the L^∞ norm). To do this, one uses the generalized pseudo-conformal transformation applied to solutions of a special nonlinear wave equation.

In Section 5.3, we make some considerations on hyperbolically radial solutions. These solutions will be defined on special regions in $\mathbb{R}^d$, over which one may observe several qualitative properties. The key ingredient is a reduction to the radial (NLS) equation, which allows to transfer results to this setting. The extension of such solutions to the whole space is a very difficult question (see Remark 5.3.3 and [46]).

In Section 5.4, we turn our study to *spatial plane wave* solutions, that is, solutions of the form $u(t,x,\mathbf{y}) = f(t, x - c\cdot\mathbf{y})$. For these solutions, one may also observe global existence and blow-up. Furthermore, we study local well-posedness on spaces that include both spatial plane waves and H^1 solutions and study the stability of these plane waves regarding small H^1 perturbations.

Finally, in Section 5.5, we consider *spatial standing waves*, that is, solutions of the form $u(t,x,\mathbf{y}) = e^{i\omega x}\phi(t,\mathbf{y})$. As in the spatial plane wave case, we study local well-posedness on spaces that include both spatial standing waves and H^1 solutions and study the H^1-stability of these spatial standing waves.

Remark 5.1.1. A motivation to consider spatial plane waves and spatial standing waves is the fact that, for some models in nonlinear optics (see [68]), the negative direction x actually represents *time* and so these solutions are such that their *time* evolution (in the physical model) is of a specific form. Actually, in this context, the term "spatial" can be a bit misleading, but we introduce it so that there is no confusion with the standard mathematical notion of plane waves and standing waves. These solutions do not present finite energy in the H^1 sense; however, they do have finite energy on the plane Oyz, transverse to the axis of propagation, for each fixed physical time x. Notice that such a question is not posed on the (NLS) model, since there is no dependence on the physical time. One could even argue that, since the energy of solutions of (NLS) has no dependence in time, the integration of the energy in the time variable is infinite. Therefore, in the context of nonlinear optics, the finite-energy assumption in both x and $\mathbf{y}$ directions may be too restrictive and should be replaced by a finite-energy condition on the transverse plane. This is supported by the difficulty in finding examples of finite-energy solutions. Notice that these solutions do exist; however, an explicit example is yet to be found.

One of the main tools in the development of a qualitative theory for a given equation is to figure out certain spatial quantities whose evolution in time has a very explicit form (in most cases, it turns out that they are, in fact, constant in time). As in (NLS), (HNLS) has an Hamiltonian formulation, which enables one to obtain conservation laws from invariances of the equation (which are, usually, more obvious to determine).

Since (HNLS) and (NLS) are quite similar in an algebraic way, one might expect that invariances of (NLS) have a very close-related counterpart in the (HNLS) setting. In fact, one has the following invariances of the (HNLS):

1. Space-time translations: $v(x, \mathbf{y}, t) = u(x + x_0, \mathbf{y} + \mathbf{y}_0, t + t_0)$;
2. Gauge invariance; $v(x, \mathbf{y}, t) = e^{i\theta} u(x, \mathbf{y}, t)$;
3. Galilean invariance: $v(x, \mathbf{y}, t) = e^{\frac{i}{2}(ax - \mathbf{b}\cdot\mathbf{y}) - \frac{i}{4}(a^2 - |\mathbf{b}|^2)t} u(x - at, \mathbf{y} - \mathbf{b}t, t)$;
4. Dilation invariance; $v(x, \mathbf{y}, t) = \lambda^{\frac{2}{\sigma}} u(\lambda x, \lambda \mathbf{y}, \lambda^2 t)$;
5. Hyperbolic invariance ($d = 2$):
$$v(x, y, t) = u(x \cosh\alpha + y \sinh\alpha, x \sinh\alpha + y \cosh\alpha, t);$$

Remark 5.1.2. Since the dilation invariance is the same for both (NLS) and (HNLS), the notion of B-criticality (see Appendix B) is the same.

The application of the technique presented in Appendix C implies the following conservation laws, which may be rigorously justified using the same process as in (NLS):

1. Conservation of energy and linear momentum:
$$\frac{dE(u(t))}{dt} := \frac{d}{dt}\left(\int \frac{|u_x(t)|^2}{2} - \frac{|\nabla_{\mathbf{y}} u(t)|^2}{2} - \frac{\lambda}{\sigma+2}|u(t)|^{\sigma+2}\right) = 0,$$
$$\frac{d}{dt} \operatorname{Im} \int \bar{u}(t) \nabla u(t) = 0;$$
2. Conservation of mass:
$$\frac{d}{dt}\int |u(t)|^2 = 0;$$
3. Center of mass evolution law:
$$\frac{d}{dt}\int x|u(t)|^2 = \operatorname{Im}\int \bar{u}(t) u_x(t), \quad \frac{d}{dt}\int y|u(t)|^2 = -\operatorname{Im}\int \bar{u}(t) u_y(t);$$
4. Virial identity (part I):
$$\frac{d}{dt}\operatorname{Im}\int \bar{u}(t)(x u_x(t) - \mathbf{y}\cdot\nabla_{\mathbf{y}} u(t)) = 4E(u_0) + \lambda\left(\frac{2d+4}{\sigma+2} - d\right)\int |u(t)|^{\sigma+2}$$

Remark 5.1.3. Notice that the energy gives no direct information on the L^2-norm of the gradient. As a consequence, several techniques that are available for the (NLS) are unusable here. For example, the standard global existence result in the L^2-subcritical case relying on Gagliardo-Nirenberg's inequality is not applicable.

In the context of (HNLS), the pseudo-conformal transform (see Appendix B) induces the conservation law

$$\frac{d}{dt}\int(|x|^2-|\mathbf{y}|^2)|u(t)|^2 = 4\,\mathrm{Im}\int \bar{u}(t)(xu_x-\mathbf{y}\cdot\nabla_{\mathbf{y}}u(t))$$

which implies the (second part of the) Virial identity

$$\frac{d^2}{dt^2}V(u(t)) := \frac{d^2}{dt^2}\int(|x|^2-|\mathbf{y}|^2)|u(t)|^2 = 16E(u_0)+4\lambda\left(\frac{2d+4}{\sigma+2}-d\right)\int|u(t)|^{\sigma+2}.$$

Remark 5.1.4. Similarly to Remark 5.1.3, note that V is not a definite functional, which overthrows, for example, the variance blow-up argument for the supercritical focusing equation. It becomes clear that there is a certain "competition" between the x-direction and the $\mathbf{y}$-direction (see [68] for a discussion on this topic). However, it is not clear at all how does this competition evolve in time: such a problem remains open.

Remark 5.1.5. As in the L^2-critical (NLS), the pseudo-conformal transform may be used to obtain global solutions of (HNLS) for small H^2 initial data and large σ. Specifically, if one considers $v=\mathcal{T}_{0,-1}u$, a straightfoward calculation shows that u is a solution of (5.1.1) on $[0,T)$ if and only if v is a solution of the corresponding nonautonomous equation

$$iv_t+v_{xx}-\Delta_{\mathbf{y}}v+\lambda(1-t)^{\frac{d\sigma-4}{2}}|v|^\sigma v=0$$

on the interval $[0,\frac{T}{1+T})$. Therefore, the global existence for u is equivalent to the existence of v on $[0,1)$, which may be achieved using Strichartz estimates (see, for example, [15]).

5.2 A family of semiclassical solutions

We apply the transformation $\mathcal{T}_{a_0,k}$ (cf. Appendix B) to obtain a family of solutions of (HNLS) in the L^2-critical case $\sigma=4/d$. The idea will be to use bound-states of (HNLS) (which are never in H^1, as proved in [30]). We say that a u is a semiclassical solution of (HNLS) if $u\in C([0,T),L^\infty(\mathbb{R}^d))$ and u satisfies (HNLS) in the distributional sense. The following theorem displays a family of semiclassical solutions as well as their dynamics.

Theorem 5.2.1. *Let $A_0\in L^\infty(\mathbb{R}^d)$ be a solution of*

$$u_{xx}-\Delta_{\boldsymbol{y}}u+\lambda|u|^{4/d}u=(k(x^2-|\boldsymbol{y}|^2)+\gamma_0)u,\ \ k,\gamma_0\in\mathbb{R}. \tag{5.2.1}$$

Then the (HNLS) admits the following family of semiclassical solutions:

$$\psi(t,x,\boldsymbol{y}) = \exp\left(-\frac{d}{2}\int_0^t a(\tau)d\tau\right) A_0\left((x,\boldsymbol{y})\exp\left(-\int_0^t a(\tau)d\tau\right)\right)$$
$$\times \exp\left(ia(t)\frac{x^2-|\boldsymbol{y}|^2}{4}\right)\exp\left(i\gamma_0\int_0^t \exp\left(-2\int_0^s a(\tau)d\tau\right)ds\right) \quad (5.2.2)$$

with $a'(t)+a(t)^2 = 4k\exp\left(-4\int_0^t a(\tau)d\tau\right)$, $a(0)=a_0\in\mathbb{R}$. *Futhermore, if* $A_0(0)\neq 0$ *and*

1. *if* $k<0$*, the solution blows up in finite time in the* L^∞ *norm, for any initial data* a_0*;*

2. *if* $k=0$*, the solution blows up at* $T=-1/a_0$*;*

3. *if* $k>0$*, the solution is global in time and its* L^∞ *norm decays like* $O(1/t)$*.*

Proof. First of all, notice that $\phi(t,x,\mathbf{y}) = e^{i\gamma_0 t}A_0(x,\mathbf{y})$ is a solution of

$$iu_t + \Box u + \lambda|u|^{4/d}u - k(x^2-|\mathbf{y}|^2)u = 0.$$

Hence, for any given $a_0\in\mathbb{R}$,

$$\psi(t,x,\mathbf{y}) = (\mathcal{T}_{a_0,k}\phi)(t,x,\mathbf{y}) = e^{i\gamma_0 g(t)}\phi\left(\frac{x}{b(t)},\frac{\mathbf{y}}{b(t)}\right)\exp\left(\frac{ia(t)(x^2-|\mathbf{y}|^2)}{4}\right)f(t)$$

is a solution of (HNLS). Using (B.3), one arrives at the expression (5.2.2).

To study the long-time behaviour of these solutions, we first observe that, since $a''+6a'a+4a^3=0$,

$$a(t) = \frac{t+c_2}{(t+c_2)^2+c_1}, \quad \text{for some } c_1,c_2\in\mathbb{R}.$$

It follows from a simple computation that $c_1 = a'(0)+a(0)^2 = 4k$. Now one splits in two possibilities:

1. if $k\leqslant 0$, then a blows up at $t_0=\sqrt{4|k|}-c_2$ as $1/t^2$ for $k<0$ and as $1/t$ for $k=0$. Since $A_0(0)\neq 0$, it follows from (5.2.2) that ψ also blows up in the L^∞ norm at time t_0;

2. if $k>0$, then a is global in time and decays as $O(1/t)$, which implies in turn that ψ is global in time and presents the same decay in L^∞.

□

Remark 5.2.1. A different technique using a hydrodynamical approach was used by O. Rozanova ([64]) to obtain a similar class of solutions for the (NLS). In fact, we observe that such a technique results in the generalized pseudo-conformal transform $\mathcal{T}_{a_0,k}$, which endows this transformation with a concrete physical interpretation.

REMARK 5.2.2. In the particular case $k = 0$, $\gamma_0 < 0$, $\lambda \geqslant 0$ and $d = 2$, we obtain easily an L^∞ solution of (5.2.1). This is an elementary consequence of the energy integral for the Klein-Gordon equation $u_{xx} - u_{yy} + \lambda |u|^2 u = \gamma_0 u$,

$$E(u(x)) = \int |u_x(x)|^2 dy + \int |u_y(x)|^2 dy + \lambda \int |u(x)|^4 dy - \gamma_0 \int |u(x)|^2 dy = E(u(0)),$$

which implies

$$\|u\|_{L^\infty(\mathbb{R}^2)} \lesssim \sup_x \|u(x, \cdot)\|_{H^1(\mathbb{R})} \leqslant E(0).$$

5.3 Hyperbolically symmetric solutions

In this section, we focus our attention on a particular class of non-integrable solutions of (5.1.1) in $d = 2$. These solutions are invariant for the hyperbolic invariance, and so, for each time $t \in [0, T)$, they are constant on the hyperbolas $x^2 - y^2 = k$. We refer these solutions as having hyperbolic symmetry. More precisely, we look for solutions of (5.1.1) of the form

$$u(t, x, y) = \begin{cases} \Phi(t, \sqrt{x^2 - y^2}) = \Phi(t, r), & r^2 = x^2 - y^2 \geqslant 0 \\ \Psi(t, \sqrt{y^2 - x^2}) = \Psi(t, s), & s^2 = y^2 - x^2 \geqslant 0 \end{cases},$$

with $\Phi, \Psi : [0, T) \times]0, \infty[\to \mathbb{C}$.

Fix now $\epsilon > 0$. First, we restrict our analysis to the region

$$D^1_\epsilon = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 > \epsilon^2\}$$

and consider the problem

$$\begin{cases} i u^\epsilon_t + \Box u^\epsilon + \lambda |u^\epsilon|^\sigma u^\epsilon = 0, & u^\epsilon = u^\epsilon(t, x, y), (x, y) \in D^1_\epsilon \\ u^\epsilon(0, x, y) = u^\epsilon_0(x, y), & (x, y) \in D^1_\epsilon \\ u^\epsilon(t, x, y) = 0, & (x, y) \in \partial D^1_\epsilon, \ t \in [0, T) \end{cases} \tag{5.3.1}$$

In the following and for simplicity of notation, we shall drop the superscript on u^ϵ. Next, we look for solutions of (5.3.1) with hyperbolic symmetry: $u(t, x, y) = \Phi(t, r)$, $r = \sqrt{x^2 - y^2} > \epsilon$. Since

$$\Box u = \Phi_{rr} + \frac{\Phi_r}{r}, \ r > \epsilon$$

the (HNLS) becomes

$$i\Phi_t + \Phi_{rr} + \frac{\Phi_r}{r} + \lambda |\Phi|^\sigma \Phi = 0$$

and if we set $\tilde{u}(t, x, y) = \Phi(t, \sqrt{x^2 + y^2})$, it follows that problem (5.3.1) is formally equivalent to the radial (NLS) problem:

$$\begin{cases} i\tilde{u}_t + \Delta \tilde{u} + \lambda |\tilde{u}|^\sigma \tilde{u} = 0, & \tilde{u} = \tilde{u}(t, x, y), (x, y) \in \Omega_\epsilon = \mathbb{R}^2 \backslash \overline{B(0, \epsilon)} \\ \tilde{u}(0, x, y) = \tilde{u}_0(x, y), & (x, y) \in \Omega_\epsilon \\ \tilde{u}(t, x, y) = 0, & (x, y) \in \partial\Omega_\epsilon, \ t \in [0, T) \end{cases}. \tag{5.3.2}$$

It becomes clear that the solutions of (5.3.1) with hyperbolic symmetry are closely related to radial solutions of the (NLS) on the exterior domain $\Omega_\epsilon = \mathbb{R}^2 \backslash \overline{B(0,\epsilon)}$. In a very similar way, setting

$$D_\epsilon^2 = \{(x,y) \in \mathbb{R}^2 : y^2 - x^2 > \epsilon^2\},$$

a solution with hyperbolic symmetry $v(t,x,y) = \Psi(t,s)$, $s = \sqrt{y^2 - x^2} > \epsilon$, of

$$\begin{cases} iv_t + \Box v + \lambda |v|^\sigma v = 0, & v = v(t,x,y), (x,y) \in D_\epsilon^2 \\ v(0,x,y) = v_0(x,y), & (x,y) \in D_\epsilon^2 \\ v(t,x,y) = 0, & (x,y) \in \partial D_\epsilon^2, \ t \in [0,T) \end{cases}$$

corresponds, through the expression $\tilde{v}(t,x,y) = \Psi(t, \sqrt{x^2+y^2})$, to a solution of

$$\begin{cases} i\tilde{v}_t - \Delta \tilde{v} + \lambda |\tilde{v}|^\sigma \tilde{v} = 0, & \tilde{v} = \tilde{v}(t,x,y), (x,y) \in \Omega_\epsilon = \mathbb{R}^2 \backslash \overline{B(0,\epsilon)} \\ \tilde{v}(0,x,y) = \tilde{v}_0(x,y), & (x,y) \in \Omega_\epsilon \\ \tilde{v}(t,x,y) = 0, & (x,y) \in \partial\Omega_\epsilon, \ t \in [0,T) \end{cases} . \tag{5.3.3}$$

Remark 5.3.1. Notice that the (NLS) in (5.3.2) and (5.3.3) concerns the focusing and defocusing cases, respectively (or vice-versa, according to the sign of λ). We shall take note of this when we consider the similar problem to (5.3.1) on the domain $D_\epsilon^1 \cup D_\epsilon^2$.

Finally, the problem on the domain $D_0^1 = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 > 0\}$ (resp. $D_0^2 = \{(x,y) \in \mathbb{R}^2 : y^2 - x^2 > 0\}$) will be considered as well, namely

$$\begin{cases} iv_t + \Box v + \lambda |v|^\sigma v = 0, & v = v(t,x,y), (x,y) \in D_0^1 \text{ (resp. } (x,y) \in D_0^2) \\ v(0,x,y) = v_0(x,y), & (x,y) \in D_\epsilon^1 \end{cases}$$

Note that here the problem of finding solutions with hyperbolic symmetry ammounts to the study of the radial (NLS) in all of $\mathbb{R}^2$. Now we can state the following results:

Theorem 5.3.1 (Solutions with hyperbolic symmetry on D_ϵ^1)**.** *Let $\tilde{u}_0 \in H_0^1(\Omega_\epsilon)$, $\Omega_\epsilon = \mathbb{R}^2 \backslash \overline{B(0,\epsilon)}$ be a radial function, $\tilde{u}_0(x,y) = \Phi_0(\rho)$, with $\rho = \sqrt{x^2+y^2}$. Then, for $0 < \sigma < \infty$, there exists $T(\Phi_0) > 0$ and a semiclassical solution with hyperbolic symmetry of* (5.3.1)

$$u \in C([0, T(\Phi_0)); L^\infty(D_\epsilon^1)),$$

with initial data $u(0,x,y) = \Phi_0(\sqrt{x^2 - y^2})$, and one has the blow-up alternative:

$$T(\Phi_0) < \infty \Rightarrow \limsup_{t \to T(\Phi_0)} \|u(t)\|_\infty = \infty.$$

In addition, let us consider the following cases:

1. *If $0 < \sigma < 4$, then $T(\Phi_0) = \infty$;*
2. *If $4 \leqslant \sigma < \infty$ and*
   (a) *$\lambda < 0$, then $T(\Phi_0) = \infty$;*

(b) $\lambda > 0$, *let*

$$E(\Phi_0) = E(\tilde{u}_0) := \frac{1}{2}\int_{\Omega_\epsilon} |\nabla \tilde{u}_0|^2 - \frac{\lambda}{\sigma+2}\int_{\Omega_\epsilon} |\tilde{u}_0|^{\sigma+2}$$

be the energy associated with (5.3.2). *Assume that* $\tilde{u}_0 \in H^2(\Omega_\epsilon) \cap L^2(\Omega_\epsilon, (|x|^2 + |y|^2)dxdy)$ *and that the energy* $E(\tilde{u}_0)$ *verifies one of the following conditions:*

i. $E(\tilde{u}_0) < 0$;

ii. $E(\tilde{u}_0) \geqslant 0$ *and, for* $\theta(x) = \frac{1}{2}|x|^2 - \epsilon^2 \log|x|$,

$$\mathrm{Im}\int_{\Omega_\epsilon} (\nabla\theta \cdot \nabla\tilde{u}_0)\overline{\tilde{u}_0} > 0, \quad \left|\mathrm{Im}\int_{\Omega_\epsilon} (\nabla\theta \cdot \nabla\tilde{u}_0)\overline{\tilde{u}_0}\right|^2 \geqslant 8E(\tilde{u}_0)\int_{\Omega_\epsilon} \theta|\tilde{u}_0|^2;$$

Then $T(\Phi_0) < \infty$.

Theorem 5.3.2 (Solutions with hyperbolic symmetry on the cone D_0^1)**.** *Let* $\tilde{u}_0 \in H^2(\mathbb{R}^2)$, *be a radial function,* $\tilde{u}_0(x,y) = \Phi_0(\rho)$, *with* $\rho = \sqrt{x^2+y^2}$. *Then, for* $0 < \sigma < \infty$, *there exists* $T(\Phi_0) > 0$ *and a semiclassical solution with hyperbolic symmetry of* (5.3.1)

$$u \in C([0, T(\Phi_0)); L^\infty(D_0^1)),$$

with initial data $u(0,x,y) = \Phi_0(\sqrt{x^2 - y^2})$, *and one has the blow-up alternative:*

$$T(\Phi_0) < \infty \Rightarrow \limsup_{t \to T(\Phi_0)} \|u(t)\|_\infty = \infty.$$

In addition, let us consider the following cases:

1. *If* $0 < \sigma < 2$, *then* $T(\Phi_0) = \infty$;

2. *If* $2 \leqslant \sigma < \infty$ *and*

   (a) $\lambda\|\tilde{u}_0\|_2^2 < 4$, *then* $T(\Phi_0) = \infty$;

   (b) $\lambda > 0$, *let*

   $$E(\Phi_0) = E(\tilde{u}_0) := \frac{1}{2}\int |\nabla\tilde{u}_0|^2 - \frac{\lambda}{\sigma+2}\int |\tilde{u}_0|^{\sigma+2}.$$

   Suppose that $\tilde{u}_0 \in L^2(\mathbb{R}^2, (|x|^2 + |y|^2)dxdy)$ *and that* $E(\tilde{u}_0)$ *verifies one of the following conditions:*

   i. $E(\tilde{u}_0) < 0$;

   ii. $E(\tilde{u}_0) \geqslant 0$ *and*

   $$\mathrm{Im}\int (x \cdot \nabla\tilde{u}_0)\overline{\tilde{u}_0} > 0, \quad \left|\mathrm{Im}\int (x \cdot \nabla\tilde{u}_0)\overline{\tilde{u}_0}\right|^2 \geqslant 4E(\tilde{u}_0)\int |x\tilde{u}_0|^2;$$

   Then $T(\Phi_0) < \infty$ *and, for* $2 \leqslant \sigma < 4$, *the blow-up is taken at the cone* $C = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$ *in the sense that*

   $$\forall \epsilon > 0 \ \liminf_{t \to T(\Phi_0)} \|u(t)\|_{L^\infty(D_0^1 \backslash D_\epsilon^1)} = \infty \tag{5.3.4}$$

Proof. We start with the proof of Theorem 5.3.1. The local existence and uniqueness of solution for the problem (5.3.2) with $\tilde{u}_0 \in H_0^1(\Omega_\epsilon)$ is a consequence of [9, Theorem 1]: there exists $T(\tilde{u}_0) = T(\Phi_0) > 0$ and a unique maximal solution $\tilde{u} \in C([0, T(\Phi_0)); H_0^1(\Omega_\epsilon))$ of the problem (5.3.2). Since $\tilde{u}_0$ is radial symmetric, it follows by uniqueness that $\tilde{u}(t)$ is also radial symmetric for all $t \in [0, T(\Phi_0))$.

On the other hand, using the classical inequality for radial functions

$$\|\tilde{u}\|_{L^\infty(\Omega_\epsilon)} \leqslant C\|\nabla\tilde{u}\|^{\frac{1}{2}}_{L^2(\Omega_\epsilon))}\|\tilde{u}\|^{\frac{1}{2}}_{L^2(\Omega_\epsilon))}. \tag{5.3.5}$$

one has $\tilde{u} \in C([0, T_{max}), L^\infty(\Omega_\epsilon))$, $\tilde{u} = \Phi(t, \rho)$. Setting $u(t, x, y) = \Phi(t, \sqrt{x^2 - y^2})$, it follows that u is a solution of (5.3.1) in the distributional sense.

We recall that $\tilde{u}$ satisfies the following conservation laws:

$$\int_{\Omega_\epsilon} |\tilde{u}(t)|^2 = \int_{\Omega_\epsilon} |\tilde{u}_0|^2, \;\; E(\tilde{u}(t)) = E(\tilde{u}_0), \;\; 0 < t < T(\Phi_0). \tag{5.3.6}$$

We derive, from (5.3.5) and (5.3.6),

$$\begin{aligned}\int_{\Omega_\epsilon} |\nabla\tilde{u}(t)|^2 &\leqslant 2E(\tilde{u}_0) + \frac{2}{\sigma+2}\|\tilde{u}_0\|^2_{L^2(\Omega_\epsilon)}\|\tilde{u}(t)\|^\sigma_{L^\infty(\Omega_\epsilon)} \\ &\leqslant 2E(\tilde{u}_0) + \frac{2}{\sigma+2}\|\tilde{u}_0\|^{2+\frac{\sigma}{2}}_{L^2(\Omega_\epsilon)}\left(\int_{\Omega_\epsilon} |\nabla\tilde{u}(t)|^2\right)^{\frac{\sigma}{4}}.\end{aligned}$$

For $\sigma < 4$, we obtain the control of the norm $\|\nabla\tilde{u}(t)\|_{L^2(\Omega_\epsilon)}$ and the solution is global. The case 2.(a) is trivial by (5.3.6). Finally, under the assumptions of case 2.(b), it follows that the maximal time of existence, T_{max}, of the solution

$$u \in C([0, T_{max}), H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)) \cap C^1((0, T_{max}), L^2(\Omega_\epsilon)),$$

is finite (cf. [45, Proposition 1.6]). We claim that this implies

$$\limsup_{t\to T_{max}} \|\tilde{u}(t)\|_{L^\infty(\Omega_\epsilon)} = \infty.$$

Indeed, if this was not true, using Duhamel's formula and Gronwall's lemma, $\|\tilde{u}(t)\|_{H^2}$ should be bounded in $[0, T_{max})$, which is absurd. Hence $T(\Phi_0) = T_{max}$ and the proof of Theorem 5.3.1 is concluded.

The same procedure is used in Theorem 5.3.2, but now we have $\mathbb{R}^2$ instead of Ω_ϵ. Hence, the proof amounts to the well-known global existence and blow-up results for (NLS) in $\mathbb{R}^2$. In the case 2(b), $2 \leqslant \sigma < 4$, we have the concentration of $\tilde{u}$ at the origin (cf. [58, Remark 3.1]) in the sense that

$$\forall\epsilon > 0 \;\; \liminf_{t\to T(\Phi_0)} \|u(t)\|_{L^\infty(\{x^2+y^2<\epsilon^2\})} = \infty,$$

which implies (5.3.4). □

Remark 5.3.2. The above results are obviously valid for the domains D_ϵ^2 and D_0^2: one must simply replace λ by $-\lambda$.

Finally, one may consider the problems

$$\begin{cases} iu_t + \Box u + \lambda |u|^\sigma u = 0, & u = u(t,x,y), (x,y) \in D_\epsilon = D_\epsilon^1 \cup D_\epsilon^2 \\ u(0,x,y) = u_0(x,y), & (x,y) \in D_\epsilon \\ u(t,x,y) = 0, & (x,y) \in \partial D_\epsilon, \ t \in [0,T) \end{cases}$$

and

$$\begin{cases} iu_t + \Box u + \lambda |u|^\sigma u = 0, & u = u(t,x,y), (x,y) \in D_0 = D_0^1 \cup D_0^2 \\ u(0,x,y) = u_0(x,y), & (x,y) \in D_0 \end{cases}$$

and build L^∞ solutions by gluing solutions on regions D_ϵ^1 and D_ϵ^2 (resp. D_0^1 and D_0^2). Over D_ϵ (resp. D_0), in the case $\sigma < 4$ (resp. $\sigma < 2$), the solutions are always global. Otherwise, we note that the sign of λ is not sufficient to guarantee global existence: if $\lambda > 0$, then the equation is focusing on regions D_0^1 and D_ϵ^1; if $\lambda < 0$, then it is focusing on D_0^2 and D_ϵ^2. In either case, one may observe finite-time blow-up.

Remark 5.3.3. Fix a continuous initial data $u_0 \in C(\mathbb{R}^2)$ with hyperbolic symmetry. Consider the radial (NLS) counterparts $\widetilde{u_0|_{D_0^1}}$ and $\widetilde{u_0|_{D_0^2}}$. Assuming these are H^1 functions, one may build a L^∞ solution u of (HNLS) on D_0 with initial data u_0. The question is wether the continuity of u_0 over $\{|y| = |x|\}$ remains valid for u (in the sense that u admits a continuous extension to $\mathbb{R}^2$). The answer is, in general, negative: take Q to be the positive radial ground-state of (NLS) in $\mathbb{R}^2$ and consider the pseudo-conformal transform of $V = e^{it}Q$,

$$W(t,x,y) = (1-t)^{-1} V\left(\frac{t}{1-t}, \frac{x}{1-t}, \frac{y}{1-t}\right) \exp\left(-i\frac{(x^2+y^2)}{4(1-t)}\right).$$

One easily checks that W is radial, $W(0,0,0) = V(0,0,0) = Q(0)$ and $|W(t,0,0)| = (1-t)^{-1}|V(t,0,0)| = (1-t)^{-1}Q(0,0)$. Then, setting

$$u_0(x,y) = \begin{cases} Q(\sqrt{x^2-y^2}), & x^2 - y^2 > 0 \\ e^{-i\frac{y^2-x^2}{4}} Q(\sqrt{y^2-x^2}), & y^2 - x^2 > 0 \end{cases},$$

the corresponding solution is given by

$$u(t,x,y) = \begin{cases} V(t, \sqrt{x^2-y^2}), & x^2 - y^2 > 0 \\ W(t, \sqrt{y^2-x^2}), & y^2 - x^2 > 0 \end{cases},$$

which is not continuous at $\{|y| = |x|\}$ for any positive time $t > 0$.

5.4 Spatial plane waves

In this section, we shall consider spatial plane waves, that is, solutions of the form $u(t,x,\mathbf{y}) = f(t, x - c\cdot\mathbf{y})$, where $c \in \mathbb{R}^{d-1}$, $c \neq 0$, is a fixed vector. For u to be a solution to the (HNLS), one must have

$$if_t + (1-|c|^2)f_{zz} + \lambda|f|^\sigma f = 0,\ f(0,z) = f_0(z) \tag{5.4.1}$$

One of the interesting properties is that the size of $|c|$ determines the nature of the equation: fix, for example, $\lambda = 1$. Then,

1. if $|c| < 1$, (5.4.1) is the focusing (NLS), which may exhibit blow-up phenomena;
2. if $|c| = 1$, one may solve explicIty (5.4.1):

$$f(t,z) = f(0,z)e^{i|f(0,z)|^\sigma t},\ t \in \mathbb{R}$$

   We observe that such solutions verify $|f(t,z)| = |f(0,z)|, \forall t, z$, which means that these solutions are *localized* (that is, their shape is not distorted by the flow of the equation);
3. if $|c| > 1$, (5.4.1) is the defocusing (NLS), for which no blow-up solutions exist.

Remark 5.4.1. Spatial plane waves also exist for the (NLS); however, the speed of the wave has no influence in the global dynamics. Furthermore, the existence of solutions with constant amplitude is a unique property of (HNLS).

Remark 5.4.2. If $d \geqslant 3$, fix $1 \leqslant n < d$. We write $\mathbf{x} = (x, \mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^{d-n} \times \mathbb{R}^{n-1}$. One may also consider solutions of the form $u(t,\mathbf{x}) = f(t, x - c\cdot\mathbf{y}_1, \mathbf{y}_2)$, $c \in \mathbb{R}^{d-n}$, which we shall call n-dimensional spatial plane waves. In this situation,

$$if_t + (1-|c|^2)f_{zz} - \Delta_{\mathbf{y}_2} f + \lambda|f|^\sigma f = 0.$$

If $|c| < 1$, then f satisfies a (HNLS)-type equation; if $|c| \geqslant 1$, then one arrives to the (NLS) equation, where one may observe global existence and finite-time blow-up, depending on the sign of λ.

5.4.1 Local existence

We now develop a suitable functional framework that includes both solutions in $H^1(\mathbb{R}^d)$ and spatial plane waves. Given $c \in \mathbb{R}^{d-1}\backslash\{0\}$, define the space of spatial plane waves

$$X_c = \left\{ u \in L^1_{loc}(\mathbb{R}^d) : \exists f \in H^2(\mathbb{R}) : u(x,\mathbf{y}) = f(x - c\cdot\mathbf{y}) \ a.e. \right\}$$

and endow it with the norm $\|u\|_{X_c} = \|f\|_{H^2}$ (we say that f is the profile of u). We observe that, up to measure zero sets, the correspondence between $u \in X_c$ and its profile is bijective: if $f, \tilde{f} : \mathbb{R} \to \mathbb{C}$ differ in a zero-measure set, then

$$g(x,\mathbf{y}) = f(x - c\cdot\mathbf{y}), \quad \tilde{g}(x,\mathbf{y}) = \tilde{f}(x - c\cdot\mathbf{y})$$

differ also in a zero-measure set (for the Lebesgue measure in $\mathbb{R}^d$).

If $u_0 \in X_c$, then the solution of

$$iu_t + \Box u = 0, u(0) = u_0$$

is given by $u(t, x, \mathbf{y}) = (U(t)f_0)(x - c \cdot \mathbf{y})$, where f_0 is the profile of u_0, and $U(t)$ is the Schrödinger group $e^{i(1-|c|^2)t\Delta}$ acting on $H^2(\mathbb{R})$ for $|c| \neq 1$, and the identity map for $|c| = 1$. It is clear that $u(t) \in X_c, \forall t$, and that

$$\|u(t)\|_{X_c} = \|U(t)f_0\|_{H^2} = \|f_0\|_{H^2} = \|u_0\|_{X_c},$$

which means that $U(t)$ induces naturally the group of isometries $e^{it\Box}$ on X_c. Set

$$E = H^1(\mathbb{R}^d) \oplus X_c.$$

Obviously, the group $e^{it\Box}$ is also defined on E, since it is simply the sum of the corresponding groups in each of the spaces. Moreover, define

$$X_c' = \left\{ u \in L^1_{loc}(\mathbb{R}^d) : \exists f \in L^2(\mathbb{R}) : u(x, \mathbf{y}) = f(x - c \cdot \mathbf{y}) \ a.e. \right\}$$

and $E' = H^{-1}(\mathbb{R}^d) \oplus X_c'$.

For a fixed $h \in \mathbb{R}^{d-1}$, define the translation operator $T_h : L^1_{loc}(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)$,

$$(T_h^c u)(x, \mathbf{y}) := u(x + c \cdot h, \mathbf{y} + h) \ a.e. \ (x, \mathbf{y}) \in \mathbb{R}^d.$$

If $\phi \in X_c'$, then

$$\phi(x, \mathbf{y}) = f(x - c \cdot \mathbf{y}) = f((x + c \cdot h) - c \cdot (\mathbf{y} + h)) = (T_h \phi)(x, \mathbf{y}), \ a.e. \ (x, \mathbf{y}) \in \mathbb{R}^d.$$

and therefore $T_h^c \phi = \phi$, $\forall \phi \in X_c'$.

Lemma 5.4.1. *One has* $H^{-1}(\mathbb{R}^d) \cap X_c' = \{0\}$.

Proof. Take $w \in H^{-1}(\mathbb{R}^d) \cap X_c'$. Then there exist $v_i \in L^2(\mathbb{R}^d)$, $i = 0, ..., d$, such that

$$w = v_0 + \sum_{i=1}^{d} (v_i)_{x_i}$$

and, for any $\Omega \subset \mathbb{R}^d$ open,

$$\|w\|_{H^{-1}(\Omega)} = \left(\sum_{i=0}^{d} \int_\Omega |v_i|^2 \right)^{1/2}.$$

Take $\Omega = \{(x, \mathbf{y}) \in \mathbb{R}^d : |x| < 1\}$. Fix $h \in \mathbb{R}^{d-1}$ such that $c \cdot h = 1$. Using the fact that $T_h^c w = w$, one has

$$\|w\|^2_{H^{-1}(\mathbb{R}^d)} = \sum_{i=0}^{d} \int_{\mathbb{R}^d} |v_i|^2 = \sum_{i=0}^{d} \sum_{m \in \mathbb{Z}} \int_\Omega |T^c_{2mh} v_i|^2 = \sum_{m \in \mathbb{Z}} \|T^c_{2mh} w\|^2_{H^{-1}(\Omega)} = \sum_{m \in \mathbb{Z}} \|w\|^2_{H^{-1}(\Omega)}.$$

Therefore one must have $\|w\|_{H^{-1}(\Omega)} = 0$ and so $\|w\|_{H^{-1}(\mathbb{R}^d)} = 0$. $\square$

Lemma 5.4.2. *For any $c_1, c_2 \in \mathbb{R}^{d-1}\backslash\{0\}$, $(H^{-1}(\mathbb{R}^d) \oplus X_{c_1}) \cap X_{c_2} = \{0\}$.*

Proof. Take $z \in (H^{-1}(\mathbb{R}^d) \oplus X_{c_1}) \cap X_{c_2}$. We write $z = w + \phi_1$, with $w \in H^{-1}(\mathbb{R}^d)$ and $\phi_1 \in X_{c_1}$. Fix $h \in \mathbb{R}^{d-1}$. Then

$$T_h^{c_2} z = z, \text{ i.e., } w - T_h^{c_2} w = -\phi_1 + T_h^{c_2}\phi_1.$$

The r.h.s. is in $H^{-1}(\mathbb{R}^d)$, while the l.h.s. is in X_{c_1}. Therefore both sides are equal to 0:

$$w = T_h^{c_2} w, \ \phi_1 = T_h^{c_2}\phi_1.$$

One now concludes that $w, \phi_1 = 0$ as in the previous proof. □

Theorem 5.4.3. *Let $1 \leqslant \sigma < 4/(d-2)^+$. For every $u_0 \in E$, there exists $T(u_0) > 0$ and a unique solution of* (5.1.1) *$u \in C([0, T(u_0)), E) \cap C^1((0, T(u_0)), E')$ which depends continuously on u_0. Also, the blow-up condition holds in the sense that*

$$\lim_{t \to T(u_0)} \|u(t)\|_E = \infty \quad \textit{if } T(u_0) < \infty.$$

Proof. On E, one actually observes a decoupling of the (HNLS) equation: writing the solution u as $v + \phi$, $v \in H^1(\mathbb{R}^d)$, $\phi \in X_c$, $\phi(x, \mathbf{y}) = f(x - c \cdot \mathbf{y})$, one has

$$iv_t + \Box v + \lambda|v + \phi|^\sigma(v + \phi) - \lambda|\phi|^\sigma\phi = -(i\phi_t + \Box\phi + \lambda|\phi|^\sigma\phi). \tag{5.4.2}$$

It is clear that the right hand side is in X_c'. We claim that the left hand side is in $H^{-1}(\mathbb{R}^d)$: the problem resides on the nonlinear part. Since $f \in H^2(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R})$, one has $\phi \in W^{1,\infty}(\mathbb{R}^d)$. Therefore

$$||v + \phi|^\sigma(v + \phi) - |\phi|^\sigma\phi| \lesssim |v| + |v|^{\sigma+1}$$

Since $v \in H^1(\mathbb{R}^d)$, $|v|^{\sigma+1} \in L^{\frac{\sigma+2}{\sigma+1}}(\mathbb{R}^d)$, the nonlinear term may be written as a sum of a L^2 function with a $L^{\frac{\sigma+2}{\sigma+1}}$ function, both of them lying in $H^{-1}(\mathbb{R}^d)$, by Sobolev's injection.

Using the fact that $H^{-1}(\mathbb{R}^d) \cap X_c' = \{0\}$ (see Lemma 5.4.1), one concludes that both sides of (5.4.2) are zero:

$$iv_t + \Box v + \lambda|v + \phi|^\sigma(v + \phi) - \lambda|\phi|^\sigma\phi = 0 \tag{5.4.3}$$
$$if_t + (1 - c^2)f_{zz} + \lambda|f|^\sigma f = 0 \tag{5.4.4}$$

Fix an initial data $u_0 = v_0 + \phi_0$, where ϕ_0 has profile f_0. For (5.4.4), one may use the H^2 local well-posedness result for the one dimensional (NLS) equation and obtain the profile f. Then, one focus on (5.4.3) and solves for v. Since the nonlinear term in (5.4.3) satisfies the estimates

$$\begin{aligned} &|(|v + \phi|^\sigma(v + \phi) - |\phi|^\sigma\phi) - (|w + \phi|^\sigma(w + \phi) - |\phi|^\sigma\phi)| \\ \lesssim &(1 + |v + \phi|^\sigma + |w + \phi|^\sigma)|v - w| \lesssim (1 + |v|^\sigma + |w|^\sigma + |\phi|^\sigma)|v - w| \end{aligned} \tag{5.4.5}$$

and

$$
\begin{aligned}
|\nabla(|v+\phi|^\sigma(v+\phi)-|\phi|^\sigma\phi)| \lesssim & (1+|v|^\sigma+|\phi|^\sigma)|\nabla v| \\
& +(|v|^{\sigma-1}+|\phi|^{\sigma-1})|v||\nabla\phi|, \qquad (5.4.6)
\end{aligned}
$$

the local existence result for (5.4.3) actually follows from the standard Kato's method (see [12, Section 4.4])applied in the (HNLS) context.

□

REMARK 5.4.3. Recall that, in the case $|c|=1$, the explicit solution of (5.4.4) is given by

$$
f(t,z)=f(0,z)e^{i|f(0,z)|^\sigma t}, \ t\in\mathbb{R},
$$

which implies that, if $f(0,\cdot)\in W^{1,\infty}(\mathbb{R})$, then $f(t,\cdot)\in W^{1,\infty}(\mathbb{R})$, for all $t>0$. Since no estimate of $e^{it\Box}$ over X_c is necessary, one may actually weaken the definition of X_c in this case, by replacing $f\in H^2(\mathbb{R})$ with $f\in W^{1,\infty}(\mathbb{R})$.

REMARK 5.4.4. Fix $d=2$ and $\sigma\geqslant 1$. We claim that one is able to extend the above local well-posedness result to $H^1(\mathbb{R}^2)\oplus X_{c_1}\oplus X_{c_2}$. In fact, the decoupling of the equation on each of the function spaces still holds (see Lemma 5.4.2): for $u=v+\phi+\psi$, $v\in H^1(\mathbb{R}^2)$, $\phi\in X_{c_1}$ with profile f and $\psi\in X_{c_2}$ with profile g, one has

$$
\begin{aligned}
& iv_t+\Box v+\lambda\left(|v+\phi+\psi|^\sigma(v+\phi+\psi)-|\phi|^\sigma\phi-|\psi|^\sigma\psi\right)=0 \qquad (5.4.7)\\
& if_t+(1-c_1^2)f_{zz}+\lambda|f|^\sigma f=0,\\
& ig_t+(1-c_2^2)g_{zz}+\lambda|g|^\sigma g=0.
\end{aligned}
$$

We point out that the nonlinear term of the first equation is truly in $H^{-1}(\mathbb{R}^2)$: note that

$$
\begin{aligned}
& ||v+\phi+\psi|^\sigma(\phi+\psi+v)-|\phi|^\sigma\phi-|\psi|^\sigma\psi| \\
\leqslant & ||v+\phi+\psi|^\sigma(\phi+\psi+v)-|\phi+\psi|^\sigma(\phi+\psi)|+||\phi+\psi|^\sigma(\phi+\psi)-|\phi|^\sigma\phi-|\psi|^\sigma\psi| \\
\lesssim & (|v|^\sigma+|\phi|^\sigma+|\psi|^\sigma)|v|+|\psi|^\sigma|\phi|+|\phi|^\sigma|\psi|.
\end{aligned}
$$

Using the fact that $\phi,\psi\in L^\infty(\mathbb{R}^2)$, the terms with $|v|$'s are treated as in the previous proof. It remains to check the last couple of terms. For example, take $|\phi|^\sigma\psi$. Then

$$
\begin{aligned}
\int_{\mathbb{R}^2}|\phi(x,y)|^{2\sigma}|\psi(x,y)|^2dxdy &= \int_{\mathbb{R}^2}|f(x-c_1y)|^{2\sigma}|g(x-c_2y)|^2dxdy \\
&= \frac{1}{|c_1-c_2|}\int_{\mathbb{R}^2}|f(w)|^{2\sigma}|g(z)|^2dwdz \\
&\leqslant \frac{1}{|c_1-c_2|}\|f\|_\infty^{2\sigma-2}\|f\|_2^2\|g\|_2^2
\end{aligned}
$$

and so $|\phi|^2\psi\in L^2(\mathbb{R}^2)\hookrightarrow H^{-1}(\mathbb{R}^2)$. In fact, (5.4.7) expresses the interaction between spatial plane waves with different velocities, which turns out to be an H^1 function.

REMARK 5.4.5. One may apply the construction made in this section to the case of n-dimensional spatial plane waves (cf. Remark 5.4.2), by replacing the profile space $H^2(\mathbb{R})$ by $H^k(\mathbb{R}^n)$, with $2k-2>n$, so that $H^k(\mathbb{R}^n)\hookrightarrow W^{1,\infty}(\mathbb{R}^n)$.

5.4.2 Stability result

The aim of this section is to prove that a large class of spatial plane waves is stable with respect to $H^1(\mathbb{R}^d)$ perturbations. The idea is to obtain a global existence result for small data in the large power case for equation (5.4.3). This is not trivial, as one may observe by analyzing (5.4.3): since there are both linear and quadratic in v, these lower order terms may disrupt the smallness of v. Moreover, it is not clear that the spatial plane wave substrate ϕ (which has both infinite mass and energy) does not increase indefinitely the $H^1(\mathbb{R}^d)$ component.

Remark 5.4.6. One may try to study the linear part of (5.4.3)

$$iv_t + \Box v + \lambda|\phi|^2 v + 2\phi \operatorname{Re} v\bar{\phi} = 0, \tag{5.4.8}$$

which is closely related to decay estimates of $L = i\Box + iV(t,x)[\cdot]$, where $V(t,x)[v] = \lambda|\phi|^2 v + 2\phi \operatorname{Re} v\bar{\phi}$. If one drops the second term in the potential, one might be able to use the results of [28], [29] to obtain such estimates. Their method is to consider approximations of the free group e^{itL} given by Feynman's path integral and requires some physical interpretation on the effect of the potential on the motion of quantum particles. However, if one does not drop the second term, this physical interpretation becomes unclear. Moreover, a simple calculation shows that (5.4.8) does not conserve the L^2 norm.

Theorem 5.4.4. *Let $\sigma \geqslant 4$, $d = 2$. Suppose that $c \in \mathbb{R}$ is such that $(|c| - 1)\lambda > 0$. Fix $\phi_0 \in X_c$ and let f_0 be its profile. If $f_0 \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}, x^2 dx) \cap W^{1,1}(\mathbb{R})$, then the plane wave ϕ of (HNLS) with initial data ϕ_0 is H^1-stable, i.e.,*

$$\forall \delta > 0\ \exists \epsilon > 0\ \|v_0\|_{H^1} < \epsilon \Rightarrow \|u - \phi\|_{L^\infty((0,\infty),H^1(\mathbb{R}^d))} < \delta,$$

where u is the (global) solution in E of (HNLS) with initial data $v_0 + \phi_0$.

Proof. Step 1. Properties of ϕ. Recall that $\phi(t,x,y) = f(t, x - cy)$, where $f \in H^2(\mathbb{R})$ is a solution of

$$if_t + (1 - c^2)f_{zz} + \lambda|f|^\sigma f = 0,\ f(0) = f_0.$$

Since $(|c| - 1)\lambda > 0$, the above equation is in the defocusing case. Therefore, f is global in $H^2(\mathbb{R})$ ([12, Theorem 5.3.1]).

By hypothesis, $f_0 \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}, x^2 dx)$ and so, since $\sigma \geqslant 4$, it follows from [12, Theorem 7.3.1] that

$$\|f(t)\|_{L^\infty} \leqslant \frac{C}{t^{1/2}}.$$

Moreover,

$$\|f(t)\|_{L^\infty} \leqslant C\|f(t)\|_{H^1} \leqslant CE(f_0).$$

Finally, we would like to estimate $\|\nabla f(t)\|_\infty$. From Duhamel's formula, one has

$$\nabla f(t) = S(t)\nabla f_0 + \int_0^t S(t-s)\nabla(|f(s)|^\sigma f(s))ds.$$

Hence, for $t > 2$,

$$\begin{aligned}
\|\nabla f(t)\|_{L^\infty} \leqslant& \|\nabla f_0\|_{L^1} + \int_0^t \frac{1}{\sqrt{t-s}} \|f(s)\|_{L^\infty}^{\sigma-1} \|f(s)\|_{L^2} \|\nabla f(s)\|_{L^2} ds \\
\lesssim& \|\nabla f_0\|_{L^1} + \int_0^1 \frac{1}{\sqrt{t-s}} E(f_0)^{\sigma-1} \|f_0\|_{L^2} E(f_0) ds \\
&+ \int_1^t \frac{1}{\sqrt{t-s}} \frac{1}{s^{\sigma-1/2}} \|f_0\|_{L^2} E(f_0) ds \\
\lesssim& \|\nabla f_0\|_{L^1} + C(f_0) \left(\int_0^1 \frac{1}{\sqrt{t-s}} ds + \int_1^t \frac{1}{\sqrt{t-s}} \frac{1}{s^{\sigma-1/2}} ds \right) \\
\lesssim& \|\nabla f_0\|_{L^1} + C(f_0) \left(\frac{1}{\sqrt{t-1}} + \int_1^{t-1} \frac{1}{s^{\sigma-1/2}} ds + \frac{1}{(t-1)^{\sigma-1/2}} \int_{t-1}^t \frac{1}{\sqrt{t-s}} \right) \\
\lesssim& \|\nabla f_0\|_{L^1} + C(f_0) \left(\frac{1}{\sqrt{t-1}} + 1 + \frac{1}{(t-1)^{\sigma-1/2}} \right) \leqslant C(f_0).
\end{aligned}$$

For $t \leqslant 2$, a similar procedure allows one to bound $\|\nabla f(t)\|_{L^\infty} \leqslant C(f_0)$. Therefore ∇f is uniformly bounded in L^∞. Since $\phi(t,x,y) = f(t, x-cy)$, it follows that

$$\|\phi(t)\|_{L^\infty(\mathbb{R}^2)} \leqslant \min\left\{ \frac{C}{t^{1/2}}, CE(f_0) \right\}, \quad \|\nabla\phi(t)\|_{L^\infty(\mathbb{R}^2)} \leqslant C(f_0).$$

Step 2. Setup. Fix $v_0 \in H^1(\mathbb{R}^2)$ and consider the corresponding solution v of

$$iv_t + \Box v + \lambda(|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi) = 0.$$

We recall that v is defined on $(0, T(u_0))$, where $u_0 = v_0 + \phi_0$. Since f is global in $H^2(\mathbb{R})$, the blow-up alternative of Theorem 5.4.3 then implies that, if $T(u_0) < \infty$,

$$\|v(t)\|_{H^1} \to \infty, \ t \to T(u_0).$$

For a given $\eta > 0$, which shall be fixed later, take $T > 0$ large enough so that

$$\|\phi(t)\|_{L^\infty} \leqslant \eta, \ t \geqslant T.$$

If $\|v_0\|_{H^1} < \epsilon$ is small enough, the fixed-point used in the local existence result implies that v is defined on $(0, T+1)$ and that $\|v(t)\|_{H^1} \leqslant 2\|v_0\|_{H^1}$, $0 < t < T+1$. Therefore, one may, without loss of generality, take as initial data $u(T) = v(T) + \phi(T)$ and prove the stability result.

Choose $\theta \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ such that

$$0 \leqslant \theta \leqslant 1, \quad \theta(w,z) \equiv 1, \text{ for } w \leqslant z, \quad \theta(w,z) \equiv 0, \text{ for } w \geqslant 2z.$$

We develop the nonlinear part as

$$|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi = \theta(|v|,|\phi|)\left(|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi\right)$$

$$\begin{aligned} &+ (1-\theta(|v|,|\phi|))\left(|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi\right) \\ &=: g_1(v,\phi) + g_2(v,\phi). \end{aligned}$$

The definition of θ implies that

$$|g_1(v,\phi)| = \theta(|v|,|\phi|)||v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi| \lesssim \theta(|v|,|\phi|)(|v|^\sigma + |\phi|^\sigma)|v| \lesssim |\phi|^\sigma|v|$$

and

$$\begin{aligned} |g_2(v,\phi)| &= (1-\theta(|v|,|\phi|))\,||v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi| \\ &\lesssim (1-\theta(|v|,|\phi|))\,(|v|^\sigma + |\phi|^\sigma)|v| \lesssim |v|^{\sigma+1}. \end{aligned}$$

In a similar fashion, we write

$$\begin{aligned} \nabla(|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi) &= \theta(|v|,|\phi|)\nabla\left(|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi\right) \\ &+ (1-\theta(|v|,|\phi|))\,\nabla\left(|v+\phi|^\sigma(v+\phi) - |\phi|^\sigma\phi\right) \\ &=: g_3(v,\phi) + g_4(v,\phi) + g_5(v,\phi) + g_6(v,\phi). \end{aligned}$$

where

$$|g_3(v,\phi)| \lesssim |v|^\sigma|\nabla\phi|, \quad |g_4(v,\phi)| \lesssim |\phi|^{\sigma-1}|\nabla\phi||v|$$
$$|g_5(v,\phi)| \lesssim |v|^\sigma|\nabla v|, \quad |g_6(v,\phi)| \lesssim |\phi|^\sigma|\nabla v|$$

Define, for $j = 1,2$, $\rho_j = \sigma + j$ and γ_j such that (γ_j,ρ_j) is an admissible pair. Consider, for $0 < t < T(u_0)$,

$$h(t) = \|v\|_{L^\infty((0,t),H^1(\mathbb{R}^2))} + \sum_{j=1}^2 \|v\|_{L^{\gamma_j}((0,t),W^{1,\rho_j}(\mathbb{R}^2))}.$$

We write Duhamel's formula,

$$v(t) = U(t)v_0 + \sum_{i=1}^2 \int_0^t U(t-s)g_i(v(s),\phi(s))ds$$

and

$$\nabla v(t) = U(t)\nabla v_0 + \sum_{i=3}^6 \int_0^t U(t-s)g_i(v(s),\phi(s))ds.$$

Therefore, for any admissible pair (q,r),

$$\|v\|_{L^q((0,t),W^{1,r}(\mathbb{R}^2))} \leqslant C\|v_0\|_{H^1} + \sum_{i=1}^6 \left\|\int_0^\cdot U(\cdot - s)g_i(v(s),\phi(s))ds\right\|_{L^q((0,t),L^r(\mathbb{R}^2))}.$$

For the sake of simplicity, we shall omit both the temporal and spatial domains. In the next steps, we shall estimate each term of the sum by a suitable power of $h(t)$. Constants depending on ϕ will be ommited.

Step 3. Estimate of higher order terms in v. Set

$$\mu_j = \frac{(\sigma+j-2)(\sigma+j)}{2}.$$

Since $\sigma \geqslant 4$, we have $\mu_j > \gamma_j$. First, we estimate the pure power terms g_2 and g_5: using Strichartz's estimates and Hölder inequality,

$$\begin{aligned}\left\|\int_0^{\cdot} U(\cdot - s)g_2(v(s),\phi(s))ds\right\|_{L^q(L^r)} &\lesssim \left\||v|^{\sigma+1}\right\|_{L^{\gamma_2'}(L^{\rho_2'})} \\ &\lesssim \|v\|^{\sigma}_{L^{\mu_2}(L^{\rho_2})}\|v\|_{L^{\gamma_2}(L^{\rho_2})} \\ &\lesssim \left(\|v\|^{\sigma}_{L^{\gamma_2}(L^{\rho_2})} + \|v\|^{\sigma}_{L^{\infty}(L^{\rho_2})}\right)\|v\|_{L^{\gamma_2}(L^{\rho_2})} \\ &\lesssim h(t)^{\sigma+1}.\end{aligned}$$

In a similar way,

$$\begin{aligned}\left\|\int_0^{\cdot} U(\cdot - s)g_5(v(s),\phi(s))ds\right\|_{L^q(L^r)} &\lesssim \left\||v|^{\sigma}|\nabla v|\right\|_{L^{\gamma_2'}(L^{\rho_2'})} \\ &\lesssim \|v\|^{\sigma}_{L^{\mu_2}(L^{\rho_2})}\|\nabla v\|_{L^{\gamma_2}(L^{\rho_2})} \lesssim h(t)^{\sigma+1}.\end{aligned}$$

Now we move to the estimate for g_3:

$$\begin{aligned}\left\|\int_0^{\cdot} U(\cdot - s)g_3(v(s),\phi(s))ds\right\|_{L^q(L^r)} &\lesssim \left\||v|^{\sigma}|\nabla \phi|\right\|_{L^{\gamma_1'}(L^{\rho_1'})} \\ &\lesssim \|v\|^{\sigma-1}_{L^{\mu_1}(L^{\rho_1})}\|\nabla v\|_{L^{\gamma_1}(L^{\rho_1})} \lesssim h(t)^{\sigma}.\end{aligned}$$

Step 4. Estimate for the linear terms in v. For $i = 1, 4, 6$, we have

$$\left\|\int_0^{\cdot} U(\cdot - s)g_i(v(s),\phi(s))ds\right\|_{L^q(L^r)} \lesssim \|g_i(v,\phi)\|_{L^1(L^2)} \lesssim \left\||\phi|^{\sigma}|v|\right\|_{L^1(H^1)}.$$

Then, using the decay estimate for $\|\phi\|_\infty$, for $b > 0$ small enough,

$$\begin{aligned}\left\||\phi|^{\sigma}|v|\right\|_{L^1(H^1)} &\lesssim \int_0^t \|\phi(s)\|_\infty^{\sigma-1}\|\phi(s)\|_{W^{1,\infty}}\|v(s)\|_{H^1}ds \\ &\lesssim \|v\|_{L^\infty(H^1)}\|\phi\|_{L^\infty(W^{1,\infty})}\|\phi\|^b_{L^\infty(L^\infty)}\int_0^t \|\phi(s)\|_\infty^{\sigma-1-b}ds \\ &\lesssim \|v\|_{L^\infty(H^1)}\eta^b\left(1 + \int_1^t \frac{1}{s^{\frac{\sigma-1-b}{2}}}ds\right) \lesssim \eta^b h(t).\end{aligned}$$

Step 5. Conclusion. Putting together Steps 2, 3 and 4, there exists a universal constant D such that

$$h(t) \leqslant D\left(\|v_0\|_{H^1} + h(t)\eta^b + h(t)^{\sigma} + h(t)^{\sigma+1}\right).$$

Now, for $\eta < (1/2D)^{1/b}$ and since $\|v_0\|_{H^1} < \epsilon$,

$$h(t) \lesssim \epsilon + \left(h(t)^\sigma + h(t)^{\sigma+1}\right)$$

If ϵ is sufficiently small, then the above inequality implies $h(t) \in [0, h_0] \cup [h_1, \infty)$, for some $\epsilon < h_0 < \delta, h_1$. Since $h(0) < \epsilon$, by continuity, one has $h(t) \leqslant h_0 < \delta$, for all $t < T(u_0)$. The blow-up alternative then implies that $T(u_0) = \infty$, which concludes the proof. □

Consider the (NLS) equation in dimension two. Fix an initial data $f_0 \in H^3(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (|x|^2 + |y|^2)dxdy) \cap W^{1,1}(\mathbb{R}^2)$. For $\sigma \geqslant 2$, [12, Theorem 6.2.1] may be adapted to prove that, if $\|f_0\|_{H^3}$ is sufficiently small, then the corresponding solution f is global and bounded in $H^3(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2)$. Moreover, one has

$$\|f(t)\|_\infty \lesssim \frac{1}{t}\|(|x| + |y|)f_0\|_2\|f_0\|_2 \text{ for } t \text{ large.}$$

This decay allows one to prove the following

Theorem 5.4.5. *Let $\sigma \geqslant 7/3$, $d = 3$. Fix a 2-dimensional spatial plane wave ϕ of (HNLS), with speed $|c| > 1$, and let f_0 be its initial profile. If $f_0 \in H^3(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (|x|^2 + |y|^2)dxdy) \cap W^{1,1}(\mathbb{R}^2)$ has sufficiently small H^3 norm, then ϕ is H^1-stable, i.e.,*

$$\forall \delta > 0 \ \exists \epsilon > 0 \ \|v_0\|_{H^1} < \epsilon \Rightarrow \|u - \phi\|_{L^\infty((0,\infty), H^1(\mathbb{R}^d))} < \delta,$$

where u is the (global) solution in E of (HNLS) with initial data $v_0 + \phi_0$.

5.5 Spatial standing waves

One of the ways to overcome the presence of a negative direction is to search for solutions of the (HNLS) of the form $u(t, x, \mathbf{y}) = e^{i\omega x}\phi(t, \mathbf{y})$, (somehow in analogy to the usual notion of bound-state - recall that in some models of nonlinear optics, these are truly time-periodic solutions). Inserting this expression into the equation,

$$i\phi_t - \omega^2\phi - \Delta_{\mathbf{y}}\phi + \lambda|\phi|^\sigma\phi = 0.$$

Setting $v(t, \mathbf{y}) = e^{-i\omega^2 t}\phi(-t, \mathbf{y})$, one arrives to

$$iv_t + \Delta_{\mathbf{y}}v - \lambda|v|^\sigma v = 0.$$

which is the (NLS) in $\mathbb{R}^{d-1}$. Consider the initial value problem

$$iv_t + \Delta_{\mathbf{y}}v - \lambda|v|^\sigma v = 0, v(0, \mathbf{y}) = v_0(\mathbf{y}) \in H^1(\mathbb{R}^{d-1}).$$

As it is well-known,

1. for $\lambda > 0$ or $\sigma < 4/(d-1)$, one has global existence of solutions in $H^1(\mathbb{R}^{d-1})$;
2. for $\lambda < 0$ and $\sigma > 4/(d-1)$, initial data $v_0 \in H^1(\mathbb{R}^{d-1}) \cap L^2(\mathbb{R}^{d-1}, |\mathbf{y}|^2 d\mathbf{y})$ with negative energy blows up in finite time.

5.5.1 Local existence and stability

As in section 5.4.1, one may build a local well-posedness theory to include both H^1 solutions and spatial standing waves. Set $k = \lfloor \frac{d+1}{2} \rfloor + 1$, where $\lfloor x \rfloor$ denotes the integer part of x. If one defines

$$Y_\omega = \left\{ \phi \in L^1_{loc}(\mathbb{R}^d) : \exists f \in H^k(\mathbb{R}^{d-1}) : \phi(x, \mathbf{y}) = e^{i\omega x} f(\mathbf{y}) \ a.e. \right\},$$

$$Y'_\omega = \left\{ \phi \in L^1_{loc}(\mathbb{R}^d) : \exists f \in H^{k-2}(\mathbb{R}^{d-1}) : \phi(x, \mathbf{y}) = e^{i\omega x} f(\mathbf{y}) \ a.e. \right\}$$

and set $F = H^1(\mathbb{R}^d) \oplus Y_\omega$ and $F' = H^{-1}(\mathbb{R}^d) \oplus Y'_\omega$, then one has the following

Theorem 5.5.1. *Let $1 \leqslant \sigma < 4/(d-2)^+$. For every $u_0 \in F$, there exists $T(u_0) > 0$ and a unique solution of* (5.1.1) *$u \in C([0, T(u_0)), F) \cap C^1((0, T(u_0)), F')$ which depends continuously on u_0. Also, the blow-up condition holds in the sense that*

$$\lim_{t \to T(u_0)} \|u(t)\|_F = \infty, \quad \text{if } T(u_0) < \infty.$$

Sketch of the proof. The proof is almost identical to that of Theorem 5.4.3. First of all, since $H^{-1}(\mathbb{R}^d) \cap Y'_\omega = \emptyset$, a solution $u = v + \phi \in F$ of (HNLS) with initial data $u_0 = v_0 + \phi_0 \in F$, $\phi_0(\mathbf{y}) = e^{i\omega x} f_0(\mathbf{y})$, is equivalent to a solution of the system

$$iv_t + \Box v + \lambda |v + \phi|^\sigma (v + \phi) - \lambda |\phi|^\sigma \phi = 0, \quad v(0) = v_0 \tag{5.5.1}$$

$$if_t + \Delta_{\mathbf{y}} f - \lambda |f|^\sigma f = 0, \quad f(0) = f_0, \quad \phi(t, x, \mathbf{y}) = e^{i\omega x - i\omega^2 t} f(-t, \mathbf{y}). \tag{5.5.2}$$

One then proceeds to solve (5.5.2) *backwards* in time using the usual H^2 local well-posedness results for (NLS). Finally, the fact that $f \in W^{1,\infty}(\mathbb{R}^2)$ and the estimates (5.4.5) and (5.4.6) allow the use of Kato's method to build the unique solution v of (5.5.1). □

Moreover, one may also derive H^1-stability for spatial standing waves, in a completely analogous fashion:

Theorem 5.5.2. *Fix $\omega \in \mathbb{R}$, $\lambda > 0$, $d = 2$ and $\sigma \geqslant 3$. Given $\phi_0 \in Y_\omega$, suppose that its profile f_0 satisfies $f_0 \in L^2(y^2 dy) \cap H^2(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$. Let ϕ be the spatial standing wave with initial data ϕ_0. Then ϕ is H^1-stable, i.e.,*

$$\forall \delta > 0 \ \exists \epsilon > 0 \ \|v_0\|_{H^1} < \epsilon \Rightarrow \|u - \phi\|_{L^\infty((0,\infty), H^1(\mathbb{R}^d))} < \delta,$$

where u is the (global) solution of (HNLS) in F with initial data $v_0 + \phi_0$.

Sketch of the proof. Setting $\phi(t, x, y) = e^{i\omega x - i\omega^2 t} f(-t, y)$, one sees that

$$\|\phi(t)\|_{W^{1,\infty}(\mathbb{R}^2)} \lesssim \|f(-t)\|_{W^{1,\infty}(\mathbb{R})}$$

and that f satisfies a defocusing (NLS) in dimension 1. As in the spatial plane wave case, this implies that ϕ is global and that

$$\|\phi(t)\|_{L^\infty} \lesssim \frac{1}{t^{1/2}}, \quad \|\nabla\phi\|_{L^\infty} \lesssim 1.$$

The proof then follows from Steps 2-5 in the proof of Theorem 5.4.4, with precisely the same estimates. □

Remark 5.5.1. Spatial standing waves are solutions which lie on $H^1(\mathbb{T} \times \mathbb{R}^{d-1})$, and so one could simply try to extend the (NLS) results over this space (see, for example, [69]). We find our approach more interesting for its novelty and because it allows to understand the effect of H^1 perturbations on these special solutions.

5.6 Further comments

Let us summarize some interesting questions that rise from this chapter:

1. The H^1 framework may not be suited for some physical models. The question then is: what is a suited framework? The spaces E and F, built from some classes of solutions, indicate that a local well-posedness theory may be presented in such a way that it includes functions without decay at infinity. Even in a mathematical perspective, the "bound-state" solutions built in [46] and [53] do not lie in H^1. It would be interesting to find local-wellposedness on more general spaces, which do not demand decay at infinity.

2. A blow-up solution in the H^1 framework is yet to be found. Our solutions never possess sufficient decay to assure integrability. An example of blow-up would be of extreme importance.

3. The construction of spaces E and F has a great capacity of generalization: let $\mathcal{Z}$ be the class of functions which have a particular shape. For a given equation, which is locally well-posed in $\mathcal{X}$, suppose that one has a class of solutions in $\mathcal{Z}$. This will imply that the profile of these solutions verifies a reduced equation, for which one may have local existence over some space $\mathcal{Y}$. If $\mathcal{Z} \cap \mathcal{X} = \emptyset$, then one should be able to prove local well-posedness on

   $$\mathcal{E} = \mathcal{X} \oplus \{u \in \mathcal{Z} : \text{ the profile of } u \text{ is in } \mathcal{Y}\}.$$

   Furthermore, this allows one to obtain a suitable functional framework to study the effect of $\mathcal{X}$-perturbations on solutions in $\mathcal{Z}$.

Chapter 6

The Ginzburg-Landau equation

In this chapter, we consider the complex Ginzburg-Landau equation

$$u_t = e^{i\theta}\Delta u + e^{i\gamma}|u|^\sigma u, \tag{CGL}$$

where

$$0 \leqslant \theta, \gamma \leqslant \frac{\pi}{2},\ 0 < \sigma < 4/(d-2)^+,\ u = u(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

From a mathematical point of view, (CGL) can be seen as an interpolation between the heat equation and the Schrödinger equation. The nature of these two equations is essentially different: in the context of the heat equation, one has regularization and maximum principles (which can then be used for comparison results), while in the Schrödinger equation, the setting is Hamiltonian, with several conservation laws, but with no gain of global differentiability and no (known) maximum principle. Consequently, even though many qualitative results are known for the (NLS) and for the nonlinear heat equation

$$u_t = \Delta u + \lambda|u|^\sigma u, \quad \lambda \in \mathbb{R},$$

it is not clear how to obtain analogous results for the (CGL). In this chapter, we shall focus on three problems:

1. Existence of bound-states: as in the (NLS), one may look for solutions of the form $u(t,x) = e^{i\omega t}\phi(x)$, and so

   $$i\omega\phi = e^{i\theta}\Delta\phi + e^{i\gamma}|\phi|^\sigma\phi.$$

   The above equation is an elliptic equation with complex coefficients. The existence of bound-states for the (CGL) is, apart from some simple perturbative cases, an open problem. The "interpolation" between a Hamiltonian system and a gradient one makes it impossible to use the general techniques from critical point theory for both Hamiltonian and gradient systems. If one tries to decompose the system in the real and imaginary parts, one finds an elliptic system of non-cooperative type for which no known results are available. On the other hand, if one tries to use topological tools (like degree theory), one stumbles in the need of an *a priori* bound for the possible solutions. This is, in general, not available[1], as one can see from the existence of indefinitely large bound-states for the (NLS).

2. Existence and stability of spatial plane waves for $\theta \neq \pi/2$ and $\sigma = 2$: as for the (HNLS), one may consider solutions of the form $u(t, x, y, z) = f(t, x - cy, z)$, $c \in \mathbb{R}$. We consider the three-dimensional case because it is the first spatial dimension for which global well-posedness for any initial data is unknown. This means that the profile of the wave exists globally (since it satisfies a Ginzburg-Landau equation in dimension two), while it is not certain that perturbations in $\mathbb{R}^3$ of the spatial plane wave will not lead to blow-up behaviour. Even though the general goal will be the same as in Section 5.4.2, the technique is fairly different, due to the presence of the diffusive term $\cos\theta\Delta u$.

3. Blow-up phenomena for the case $\theta = \pi/2, \gamma \neq \pi/2$: in this last section, we show that, when $\sigma < 2/d$, any nonzero solution blows-up in finite or infinite time. We observe that, for $\sigma > 2/d$, one may prove the existence of globally small bounded solutions as in the (NLS) case.

6.1 Bound-states: existence results

To stress the difficulty in finding bound-states, we start with the following non-existence result:

Proposition 6.1.1. *Given Ω an open subset of $\mathbb{R}^d$, suppose that $u \in H_0^1(\Omega)$ is a solution of*

$$i\omega u = e^{i\theta}\Delta u + e^{i\gamma}|u|^\sigma u, \quad \omega > 0, 0 \leqslant \gamma \leqslant \theta \leqslant \pi/2,\ \gamma \neq \pi/2.$$

Then $u \equiv 0$. The same conclusion is valid if $\omega = 0$ and $\theta \neq \gamma$.

Proof. Multiply the equation by $\bar{u}$ and integrate:

$$\omega\|u\|_2^2 + \sin\theta\|\nabla u\|_2^2 = \sin\gamma\|u\|_{\sigma+2}^{\sigma+2}, \quad \cos\gamma\|\nabla u\|_2^2 = \cos\gamma\|u\|_{\sigma+2}^{\sigma+2}.$$

Then one has $\omega\cos\theta\|u\|_2^2 = \sin(\gamma - \theta)\|u\|_{\sigma+2}^{\sigma+2}$. If $u \not\equiv 0$, then either $\theta = \pi/2$ and

$$\sin(\gamma - \theta) = 0$$

or $\theta < \pi/2$ and

$$\sin(\gamma - \theta) > 0.$$

From the assumptions on γ and θ, both cases are impossible. □

[1] Using the arguments of [3], one may find an *a priori* bound under the assumption that the bound-state has a bounded Morse index.

As a consequence of the non-existence result, one can argue that a direct perturbative argument around the (NLS) ground-state is impossible. In fact, the (NLS) case corresponds to $\theta = \gamma = \pi/2$ and $\omega = 1$, which lies in the frontier of the region of non-existence. If one could apply a direct perturbative argument (that is, without having a dependence between γ and θ), one would find an open region of existence around $\theta = \gamma = \pi/2$, which is absurd.

We prove a perturbation result around a simple eigenvalue λ of the Laplace-Dirichlet operator to show existence of bound-states for the (CGL) with a linear damping term:

$$u_t = e^{i\theta}\Delta u + e^{i\gamma}|u|^\sigma u + ku, \quad k \in \mathbb{R}.$$

The result we present here is quite similar to that of [14]: therein, it is proven that, over open bounded connected subsets of $\mathbb{R}^d$, the equation

$$i\omega u = e^{i\theta}\Delta u + e^{i\gamma}|u|^\sigma u + ku$$

has a solution $(\omega, u) \in \mathbb{R} \times H_0^1(\Omega)$ if σ is sufficiently small and $(\lambda \cos\theta - k)\cos\gamma > 0$. The goal of our result is to trade the freedom in k for the freedom in σ.

Define $H := \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$. This space is a real Hilbert space when equipped with the scalar product

$$(u, v) = \mathrm{Re}\int_\Omega u\bar{v} + \mathrm{Re}\int_\Omega \Delta u \Delta\bar{v}.$$

Let λ be an eigenvalue of $-\Delta : H \mapsto L^2(\Omega)$. Assuming that the corresponding eigenspace is of the form $\mathbb{C}\phi$, we set

$$H = \mathbb{C}\phi \oplus H_1, \quad H_1 = (\mathbb{C}\phi)^\perp.$$

Theorem 6.1.2. *Suppose that Ω is a bounded, connected, open subset of $\mathbb{R}^d$. Fix $\theta, \gamma \in \mathbb{R}$ and $\sigma > 0$. Then there exist $\mu_0 > 0$ and C^1 mappings*

$$v : (-\mu_0, \mu_0) \mapsto H, \quad \omega, k : (-\mu_0, \mu_0) \mapsto \mathbb{R}$$

such that $v(0) = \phi$, $\omega(0) = -\lambda\sin\theta$, $k(0) = \lambda\cos\theta$ and

$$\Delta v + \mu e^{i(\gamma-\theta)}|v|^\sigma v + (k - i\omega)e^{-i\theta}v = 0.$$

Proof. Define the mapping $F : \mathbb{R} \times H_1 \times \mathbb{R} \times \mathbb{R} \mapsto L^2(\Omega)$ as

$$F(\mu, \zeta, \omega, k) = \Delta v + \mu e^{i(\gamma-\theta)}|v|^\sigma v + (k - i\omega)e^{-i\theta}v, \quad v = \phi + \zeta. \tag{6.1.1}$$

If one sets

$$k_0 - i\omega_0 = \lambda e^{i\theta},$$

it follows that $F(0, 0, \omega_0, k_0) = 0$. Furthermore, the mapping $(\zeta, \omega, k) \mapsto F(\mu, \zeta, \omega, k)$ is of class C^1 and

$$\frac{\partial F}{\partial \zeta}(\mu, \zeta, \omega, k)w = \Delta w + \mu e^{i(\gamma-\theta)}\left(|v|^\sigma w + \sigma|v|^{\sigma-2}v\,\mathrm{Re}(\bar{v}w)\right) + (k - i\omega)e^{-i\theta}w,$$

$$\frac{\partial F}{\partial \omega}(\mu, \zeta, \omega, k) = -ie^{-i\theta}v, \quad \frac{\partial F}{\partial k}(\mu, \zeta, \omega, k) = e^{-i\theta}v.$$

Now we check that the jacobian

$$J = \frac{\partial F}{\partial(\zeta, \omega, k)}(0, 0, \omega_0, k_0) : H_1 \times \mathbb{R} \times \mathbb{R} \mapsto L^2(\Omega)$$

is a bijection. Applying to an element $(w, a, b) \in H_1 \times \mathbb{R} \times \mathbb{R}$, we have

$$J(w, a, b) = \Delta w + \lambda w + e^{-i\theta}(b - ia)\phi.$$

If $J(w, a, b) = 0$, then

$$0 = \int J(w, a, b)\bar{\phi} = e^{-i\theta}(b - ia)\|\phi\|_2^2,$$

which implies $a, b = 0$. Thus $-\Delta w = \lambda w$ and so w is an eigenvector with eigenvalue λ. However, since $w \in H_1$, this means that $w = 0$. Hence J is injective. On the other hand, given $f \in L^2(\Omega)$, write

$$f = -e^{-i\theta}(\tilde{b} - i\tilde{a})\phi + \psi, \quad \int_\Omega \psi\bar{\phi} = 0.$$

The orthogonality condition implies that there exists $\tilde{w} \in H_1$ such that $\Delta\tilde{w} + \lambda\tilde{w} = \psi$. Then

$$J(\tilde{w}, \tilde{a}, \tilde{b}) = f,$$

which shows that J is surjective.

With the above considerations, one may apply the Implicit Function Theorem [78, Theorem 4.B] and the proof is finished. □

Corollary 6.1.3. *Suppose that Ω is a bounded, connected, open subset of $\mathbb{R}^d$ such that the Laplace-Dirichlet operator over Ω has a simple eigenvalue. Fix $0 \leqslant \theta, \gamma \leqslant \pi/2$, $\gamma \neq \pi/2$ and $\sigma > 0$. There exists $\epsilon > 0$ such that, for any k with $0 < \lambda\cos\theta - k < \epsilon$, there exist $\omega > 0$ and a solution $u \in H$ of*

$$i\omega u = e^{i\theta}\Delta u + e^{i\gamma}|u|^\sigma u + ku. \tag{6.1.2}$$

Proof. Consider the mappings v, ω, k from the previous theorem and the mapping F as in (6.1.1). Then

$$F(\mu, \zeta(\mu), \omega(\mu), k(\mu)) = 0, \quad \mu \in (-\mu_0, \mu_0).$$

Differentiating with respect to μ at $\mu = 0$, we obtain

$$e^{i(\gamma-\theta)}|\phi|^\sigma\phi + \Delta w + \lambda w + e^{-i\theta}\left(\frac{\partial k}{\partial \mu} - i\frac{\partial \omega}{\partial \mu}\right)\phi = 0, \quad w = \frac{\partial v}{\partial \mu}.$$

Multiplying by $\bar{\phi}$ and integrating over Ω, we arrive at

$$\frac{\partial k}{\partial \mu} = -\cos\gamma\frac{\|\phi\|_{\sigma+2}^{\sigma+2}}{\|\phi\|_2^2} < 0, \quad \frac{\partial \omega}{\partial \mu} = \sin\gamma\frac{\|\phi\|_{\sigma+2}^{\sigma+2}}{\|\phi\|_2^2}.$$

Thus the mapping $\mu \mapsto k(\mu)$ is locally invertible at 0, which implies that one may write $\mu = \mu(k)$, $v = v(k)$ and $\omega = \omega(k)$, for $|k - \lambda \cos\theta| < \epsilon$. Finally, if $\lambda\cos\theta - k > 0$, then $\mu(k) > 0$ and so $u(k) = \mu(k)^{\frac{1}{\sigma}} v(k)$ satisfies (6.1.2). $\square$

We end this section with an explicit computation of some bound-states in the special case $\Omega = \mathbb{R}$. First, let us state two simple facts:

- Given a positive function $a \in C^2(\mathbb{R})$ and $k \in \mathbb{R}$, writing $u(x) := a(x)\exp(ik \ln a(x))$, one has

$$u''(x) = \left(a''(x)(1+ik) + ik(1+ik)\frac{(a'(x))^2}{a(x)}\right)\exp(ik\ln a(x)).$$

- If a is the unique $H^1(\mathbb{R})$ solution of $a'' = \alpha a - \beta a^{\sigma+1}$, $\alpha, \beta > 0$, then

$$\frac{(a')^2}{a} = \alpha a - \frac{2\beta}{\sigma+2}a^{\sigma+1}.$$

  Hence both a'' and $(a')^2/a$ are linear combinations of a and $a^{\sigma+1}$.

Using these considerations, let us search for a solution $u \in H^1(\mathbb{R})$ of

$$u'' = e^{i\tilde{\theta}}u - e^{i\tilde{\gamma}}|u|^\sigma u, \quad \tilde{\theta} = \frac{\pi}{2} - \theta,\ \tilde{\gamma} = \gamma - \theta$$

of the form $u = a\exp(ik\ln a)$, where $a > 0$. Then

$$a'' - k^2\frac{(a')^2}{a} = \cos\tilde{\theta}a - \cos\tilde{\gamma}a^{\sigma+1}$$

$$ka'' + k\frac{(a')^2}{a} = \sin\tilde{\theta}a - \sin\tilde{\gamma}a^{\sigma+1}.$$

Simplifying,

$$a'' = \frac{1}{1+k^2}\left[(k\sin\tilde{\theta} + \cos\tilde{\theta})a - (k\sin\tilde{\gamma} + \cos\tilde{\gamma})a^{\sigma+1}\right] \tag{6.1.3}$$

$$\frac{(a')^2}{a} = \frac{1}{1+k^2}\left[\left(\frac{\sin\tilde{\theta}}{k} - \cos\tilde{\theta}\right)a - \left(\frac{\sin\tilde{\gamma}}{k} - \cos\tilde{\gamma}\right)a^{\sigma+1}\right] \tag{6.1.4}$$

Hence, writing

$$\alpha = \frac{k\sin\tilde{\theta} + \cos\tilde{\theta}}{1+k^2}, \quad \beta = \frac{k\sin\gamma + \cos\gamma}{1+k^2}$$

we require that

$$\frac{\sin\tilde{\theta}}{k} - \cos\tilde{\theta} = (1+k^2)\alpha, \quad \frac{\sin\tilde{\gamma}}{k} - \cos\tilde{\gamma} = (1+k^2)\frac{2\beta}{\sigma+2}. \tag{6.1.5}$$

Indeed, if these equalities hold, then (6.1.4) follows from (6.1.3), which is simply the equation for a bound-state of (NLS). The first condition in (6.1.5) implies that $\tilde{\theta} \neq 0$ and

$$k = \frac{\pm 1 - \cos\tilde{\theta}}{\sin\tilde{\theta}} =: k_{\pm}.$$

However, if $k = k_-$, then a would be a bound-state of (NLS) with negative frequency α, which does not exist (see [5]). Thus $k = k_+$ and the second condition in (6.1.5), after some simplifications, turns out to be equivalent to

$$\frac{\cos\gamma}{\sin(\gamma-\theta)} = \frac{\sin(\tilde{\theta}-\tilde{\gamma})}{\sin\tilde{\gamma}} = \frac{\sigma}{\sigma+4}.$$

A necessary condition for this equality to hold is $\gamma > \theta$, which agrees with Proposition 6.1.1. Moreover, we see that the existence of such solutions depends on a particular relationship between θ, γ and σ, which confirms somehow the perturbative results (where one parameter is always dependent on the remaining ones). Finally, we remark that, for $\gamma \leqslant \pi/4$, the condition above is never verified:

$$\frac{\cos\gamma}{\sin(\gamma-\theta)} > \frac{\cos\gamma}{\sin\gamma} \geqslant 1 > \frac{\sigma}{\sigma+4}.$$

6.2 Spatial plane waves: existence and stability

In this section, we focus on the existence and stability of spatial plane waves (cf. Section 5.4) for the (CGL) when $\sigma = 2$ and $\theta \neq \pi/2$ (that is, with a diffusion term). Before we proceed, we state some known properties of the (CGL) (see [33], [34]):

- the linear semigroup $\{T(t)\}_{t\geqslant 0}$ for (CGL) can be seen to be a convolution with a kernel which shares the properties of the heat kernel. Using the explicit formula, one may prove the decay estimate

$$\|T(t)u\|_{L^p(\mathbb{R}^d)} \lesssim t^{-\frac{1}{2}\left(\frac{d}{q}-\frac{d}{p}\right)}\|u\|_{L^q(\mathbb{R}^d)}, \quad 1 \leqslant q \leqslant p \leqslant \infty.$$

  Moreover, one has $\|T(t)u_0\|_2 \to 0$ as $t \to \infty$ (this is easily seen on the Fourier side);
- for $d = 2$, one has global well-posedness for

$$u_0 \in L^2(\mathbb{R}^2),\ \text{giving } u \in C([0,\infty); L^2(\mathbb{R}^2)) \cap C((0,\infty); L^\infty(\mathbb{R}^2))$$

  or

$$u_0 \in H^2(\mathbb{R}^2),\ \text{giving } u \in C([0,\infty); H^2(\mathbb{R}^2)).$$

- for $d = 3$ and small viscosity $\theta \sim \pi/2$, global well-posedness for generic initial data is unknown. In this case, one might expect that the Schrödinger part of the equation (which is focusing) may give birth to the formation of singularities (see [63] for strong numerical evidence).

- the nonlinearity $|u|^2u$ presents $L^d(\mathbb{R}^d)$-critical decay behaviour:

$$\||T(t)u_0|^2 T(t)u_0\|_{L^d(\mathbb{R}^d)} \sim \frac{1}{t}.$$

  Using the arguments of [44], one may prove global existence for small data in $L^d(\mathbb{R}^d)$ (this was done for the Navier-Stokes equation, which shares many similarities with the (CGL)).

With these properties in mind, we focus on the case $d = 3$. The global existence of spatial plane waves is a trivial matter: indeed, when one seeks solutions of (CGL) of the form $\phi(t, x, y, z) = f(t, x - cy, z)$, $c \in \mathbb{R}$, it is immediate that the profile f should satisfy

$$f_t = (1 + c^2)e^{i\theta}\Delta f + e^{i\gamma}|f|^2 f, \quad f = f(w, z),\ (w, z) \in \mathbb{R}^2, \tag{6.2.1}$$

which is simply (CGL) in dimension $d = 2$ (up to a scaling). Thus, given $f_0 \in H^2(\mathbb{R}^2)$, there exists a unique global solution f of (6.2.1) with initial data f_0. To fix ideas, define

$$X_c = \left\{\phi \in L^1_{loc}(\mathbb{R}^3) : \phi(x, y, z) = f\left(\frac{x - cy}{\sqrt{1 + c^2}}, z\right) \text{ a.e., } f \in H^2(\mathbb{R}^2)\right\},$$

which is a Banach space with the induced norm $\|\phi\|_{X_c} = \|f\|_{H^2}$. The semigroup $\{T(t)\}_{t\geqslant 0}$ can be extended to X_c by setting

$$(T(t)\phi)(x, y, z) := (T(t)f)\left(\frac{x - cy}{\sqrt{1 + c^2}}, z\right), \quad \text{a.e. } (x, y, z) \in \mathbb{R}^3.$$

With this extension, we say that $\phi \in C([0, \infty); X_c)$ is a solution of (CGL) if it satisfies the Duhamel formula. Thus we have

Proposition 6.2.1. *Given $\phi_0 \in X_c$, there exists a unique global solution ϕ of (CGL) in $C([0, \infty); X_c)$.*

To prove stability of spatial plane waves with respect to localized perturbations (that is, with spatial decay at infinity), one must prove a type of "small data global existence" result for the perturbed equation

$$v_t = e^{i\theta}\Delta v + e^{i\gamma}\left(|v + \phi|^2(v + \phi) - |\phi|^2\phi\right), \quad v = v(x, y, z),\ (x, y, z) \in \mathbb{R}^3. \tag{6.2.2}$$

Since the (CGL) presents global existence for small data in $L^3(\mathbb{R}^3)$, we shall assume that the initial perturbation v_0 is in $L^3(\mathbb{R}^3)$.

Lemma 6.2.2. *Fix $f_0 \in H^2(\mathbb{R}^2)$ and let f be the (global) solution of (CGL) with initial data f_0. Then*

1. *$\|f(t)\|_2 \to 0$ as $t \to \infty$;*
2. *For $\tau > 0$ sufficiently large, one has, for $2 \leqslant p \leqslant \infty$,*

$$\|f(t)\|_p \lesssim \frac{1}{(t - \tau)^{1/2 - 1/p}}\|f(\tau)\|_2, \quad t > \tau.$$

Proof. For the first part, one has

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|_2^2 = -\cos\theta\|\nabla f(t)\|_2^2 - \cos\gamma\|f(t)\|_4^4.$$

This implies that

$$\lim_{\tau\to\infty}\int_\tau^\infty \|\nabla f(t)\|_2^2 + \|f(t)\|_4^4 dt \lesssim \lim_{\tau\to\infty}\int_\tau^\infty \|\nabla f(t)\|_2^2 + \|f_0\|_2^2\|\nabla f(t)\|_2^2 dt = 0.$$

Given $\delta > 0$, there exist $t, \tau > 0$ such that

$$\begin{aligned}\|f(t)\|_2^2 &\lesssim \|T(t)f(\tau)\|_2^2 + \int_\tau^t \|f(s)\|_6^3 ds \lesssim \|T(t)f(\tau)\|_2^2 + \int_\tau^t \|\nabla f(s)\|_2^2 + \|f(s)\|_4^4 ds\\ &\lesssim \|T(t)f(\tau)\|_2^2 + \delta \lesssim 2\delta,\end{aligned}$$

since the linear semigroup satisfies $\|T(t)f(\tau)\|_2^2 \to 0$ as $t \to \infty$. For the second part, it follows from [34] that, if $\|g_0\|_2$ is sufficiently small, then the corresponding solution g satisfies

$$\|g(t)\|_p \lesssim \frac{1}{t^{1/2-1/p}}\|g_0\|_2, \quad t > 0,\ 2 \leqslant p \leqslant \infty.$$

Thus the conclusion follows from the first part, by taking $g_0 = f(\tau)$, τ large. □

Theorem 6.2.3. *Given $\phi_0 \in X_c$, let ϕ be the solution of (CGL) with initial data ϕ_0. If $v_0 \in L^3(\mathbb{R}^3)$ satisfies $\|v_0\|_3 < \epsilon$, $\epsilon = \epsilon(\phi)$ small, then there exists a unique global solution u of (CGL) with initial data $\phi_0 + v_0$ such that*

$$t^{\frac{1}{2}-\frac{3}{2p}}(u-\phi) \in C([0,\infty); L^p(\mathbb{R}^3)), \quad p = 3, \infty.$$

Moreover, one has, for $p = 3, \infty$,

$$\sup_{t>0}\left\{t^{\frac{1}{2}-\frac{3}{2p}}\|u-\phi\|_p\right\} < \delta(\epsilon), \text{ with } \delta(\epsilon) \xrightarrow[\epsilon\to 0]{} 0.$$

Proof. As it was previously discussed, the existence of u is equivalent to the existence of the remainder v solution of (6.2.2) with initial data v_0. From Lemma 6.2.2, given any $\eta > 0$, there exists $\tau > 0$ such that the profile f of ϕ satisfies

$$\|f(\tau)\|_2 < \eta, \quad \|f(t)\|_p \lesssim \frac{\eta}{(t-\tau)^{1/2-1/p}}, \quad t > \tau,\ 2 \leqslant p \leqslant \infty.$$

Moreover, since f is a global solution in H^2, we have

$$\sup_{t\in[0,\tau]}\|f(t)\|_\infty \lesssim \sup_{t\in[0,\tau]}\|f(t)\|_{H^2} \leqslant K.$$

The proof follows in two steps: first, we focus on the time interval $(0,\tau)$; then, we take as starting point $t = \tau$ and show global existence. The only difference between the two steps is the way we estimate ϕ.

Step 1. Fix $L > 0$ to be chosen later and consider the space

$$\mathcal{E} = \{v \in C([0,\tau), L^3(\mathbb{R}^3)) : \|v(t)\|_3 + t^{\frac{1}{2}}\|v(t)\|_\infty \leqslant Me^{Lt},\ 0 < t < \tau\}.$$

endowed with the distance

$$d(v,w) = \sup_{t\in(0,\tau)} \{e^{-Lt}t^{\frac{1}{2}}\|v(t) - w(t)\|_\infty + e^{-Lt}\|v(t) - w(t)\|_3\}.$$

For any $v \in \mathcal{E}$, set

$$\Phi(v)(t) = T(t)v_0 + e^{i\gamma}\int_0^t T(t-s)\left(|v+\phi|^2(v+\phi) - |\phi|^2\phi\right)(s)ds.$$

We shall prove that, for suitable choices of M and ϵ, the mapping $\Phi : \mathcal{E} \mapsto \mathcal{E}$ is a strict contraction, thus yielding a solution on the interval $(0,\tau)$. For any $0 \leqslant t \leqslant \tau$,

$$\begin{aligned}
\|\Phi(v)(t)\|_3 &\lesssim \|v_0\|_3 + \int_0^t \left(\frac{1}{(t-s)^{1/4}}\||v|^2v\|_2 + \||v|^2|\phi|\|_3 + \||\phi|^2|v|\|_3\right)ds \\
&\lesssim \|v_0\|_3 + \int_0^t \left(\frac{1}{(t-s)^{1/4}}\|v\|_\infty^{3/2}\|v\|_3^{3/2} + \|v\|_\infty\|\phi\|_\infty\|v\|_3 + \|\phi\|_\infty^2\|v\|_3\right)ds \\
&\lesssim \|v_0\|_3 + \int_0^t \left(\frac{1}{(t-s)^{1/4}}\frac{e^{3Ls}}{s^{3/4}}M^3 + \frac{e^{2Ls}}{s^{1/2}}KM^2 + e^{Ls}K^2M\right)ds \\
&\lesssim e^{Lt}\left(\epsilon + e^{2L\tau}M^3 + e^{L\tau}\tau^{1/2}KM^2 + \frac{K^2M}{L}\right)
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi(v)(t)\|_\infty &\lesssim \frac{1}{t^{1/2}}\|v_0\|_3 + \int_0^t \left(\frac{1}{(t-s)^{3/4}}\||v|^2v\|_2 + \frac{1}{(t-s)^{1/2}}(\||v|^2|\phi|\|_3 + \||\phi|^2|v|\|_3)\right)ds \\
&\lesssim \frac{1}{t^{1/2}}\|v_0\|_3 + \int_0^t \frac{1}{(t-s)^{3/4}}\frac{e^{3Ls}}{s^{3/4}}M^3ds \\
&\quad + \int_0^t \frac{1}{(t-s)^{1/2}}\left(\frac{e^{2Ls}}{s^{1/2}}KM^2 + e^{Ls}K^2M\right)ds \\
&\lesssim \frac{e^{Lt}}{t^{1/2}}\left(\epsilon + e^{2L\tau}M^3 + e^{L\tau}\tau^{1/2}KM^2 + \frac{K^2M}{(4L)^{1/4}}\tau^{3/4}\right),
\end{aligned}$$

where we used

$$\int_0^t \frac{e^{Ls}}{(t-s)^{1/2}}ds \leqslant \left(\int_0^t e^{4Ls}ds\right)^{1/4}\left(\int_0^t \frac{1}{(t-s)^{2/3}}\right)^{3/4} \lesssim \frac{e^{Lt}}{(4L)^{1/4}}t^{3/4-1/2}.$$

In a similar fashion,

$$\|\Phi(v)(t) - \Phi(w)(t)\|_3 \lesssim e^{Lt}\left(e^{2L\tau}M^2 + e^{L\tau}\tau^{1/2}KM + \frac{K^2}{L}\right)d(v,w)$$

and

$$\|\Phi(v)(t) - \Phi(w)(t)\|_\infty \lesssim t^{1/2}e^{Lt}\left(e^{2L\tau}M^2 + e^{L\tau}\tau^{1/2}KM + \frac{K^2}{(4L)^{1/4}}\tau^{3/4}\right)d(v,w).$$

Therefore, choosing $M = 2\epsilon$, for ϵ small and L large, one has

$$e^{-Lt}\|\Phi(v(t))\|_3 + t^{1/2}e^{-Lt}\|\Phi(v(t))\|_\infty < 2\epsilon, \quad d(\Phi(v),\Phi(w)) < \frac{1}{2}d(v,w)$$

and so $\Phi : \mathcal{E} \mapsto \mathcal{E}$ is a strict contraction. Thus it follows from the Banach fixed point theorem that there exists a solution v of (6.2.2) on $(0,\tau)$ with $\|v(\tau)\|_3 < 2\epsilon e^{L\tau}$.

Step 2. We now take as a starting point $t = \tau$ and use, once again, a fixed-point argument on (τ,∞). To simplify notations and without loss of generality, we assume $\tau = 0$. We now set

$$\mathcal{F} = \{v \in C([0,\infty), L^3(\mathbb{R}^3)) : \|v(t)\|_3 + t^{\frac{1}{2}}\|v(t)\|_\infty \leqslant M,\ 0 < t < \tau\}.$$

endowed with the distance

$$d(v,w) = \sup_{t\in(0,\infty)} \{t^{\frac{1}{2}}\|v(t) - w(t)\|_\infty + \|v(t) - w(t)\|_3\}$$

and show that $\Phi : \mathcal{F} \mapsto \mathcal{F}$ is a strict contraction. For convenience, we write

$$|v+\phi|^2(v+\phi) - |\phi|^2\phi = |v|^2v + g_2(v,\phi) + g_1(v,\phi),$$

where g_j, $j = 1, 2$, is essentially $|v|^j|\phi|^{3-j}$. Then

$$\|\Phi(v)(t)\|_3 \lesssim \|v_0\|_3 + \int_0^t \left(\|S(t-s)|v|^2v\|_3 + \|S(t-s)g_2(v,\phi)\|_3 + \|S(t-s)g_1(v,\phi)\|_3\right) ds.$$

We estimate each term seperately. For the cubic term in v,

$$\int_0^t \|S(t-s)|v|^2v\|_3 ds \lesssim \int_0^t \frac{1}{(t-s)^{1/4}}\||v|^2v\|_2 ds \lesssim \left(\int_0^t \frac{1}{(t-s)^{1/4}}\frac{1}{s^{3/4}}ds\right)M^3 \lesssim M^3.$$

The quadratic term is estimated as

$$\begin{aligned}\int_0^t \|S(t-s)g_2(v,\phi)\|_3 ds &\lesssim \int_0^t \frac{1}{(t-s)^{1/4}}\||v|^2|\phi|\|_2 ds\\ &\lesssim \int_0^t \frac{1}{(t-s)^{1/4}}\|\phi\|_\infty\|v\|_\infty^{1/2}\|v\|_3^{3/2} ds\\ &\lesssim \int_0^t \left(\frac{1}{(t-s)^{1/4}}\frac{1}{s^{3/4}}ds\right)M^2\eta \lesssim M^2\eta.\end{aligned}$$

For the linear term in v, one must use mixed Lebesgue estimates for the heat kernel: indeed, using the explicit formula of the semigroup, one may prove that

$$\|S(t)w\|_{L^3(\mathbb{R}^3)} \lesssim \frac{1}{t^{1/6}}\|w\|_{L^3(\mathbb{R},L^2(\mathbb{R}^2))}, \quad t > 0, w \in C_0^\infty(\mathbb{R}^3).$$

Thus

$$\begin{aligned}\int_0^t \|S(t-s)g_2(v,\phi)\|_3 ds &\lesssim \int_0^t \frac{1}{(t-s)^{1/6}} \||v||\phi|^2\|_{L^3_x(L^2_{y,z})} ds \\ &\lesssim \int_0^t \frac{1}{(t-s)^{1/6}} \left\| \|f\|^2_{L^{12}_{w,z}} \|v\|_{L^3_{y,z}} \right\|_{L^3_x} ds \\ &\lesssim \left(\int_0^t \frac{1}{(t-s)^{1/6}} \frac{1}{s^{5/6}} \|v\|_3 ds \right) \eta^2 \lesssim M\eta^2.\end{aligned}$$

Putting these estimates together, we obtain

$$\|\Phi(v)(t)\|_3 \lesssim 2\epsilon + (M^2+\eta^2)M.$$

The L^∞ estimate is obtained in a similar fashion:

$$\begin{aligned}\|\Phi(v)(t)\|_\infty &\lesssim \frac{1}{t^{1/2}}\|v_0\|_3 + \int_0^t \frac{1}{(t-s)^{3/4}} \left(\||v|^2 v\|_2 + \||v|^2|\phi|\|_2 + \left\||\phi|^2|v|\right\|_{L^\infty_x(L^{4/3}_{y,z})} \right) ds \\ &\lesssim \frac{1}{t^{1/2}}\|v_0\|_3 + \int_0^t \frac{1}{(t-s)^{3/4}} \|v\|_\infty^{3/2}\|v\|_3^{3/2} ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{3/4}} \left(\|v\|_\infty^{1/2}\|\phi\|_\infty \|v\|_3^{3/2} + \|v\|_\infty \|f\|^2_{L^{8/3}} \right) ds \\ &\lesssim \frac{1}{t^{1/2}}\|v_0\|_3 + \left(\int_0^t \frac{1}{(t-s)^{3/4}} \frac{1}{s^{3/4}} ds \right) (M^2+\eta^2)M \\ &\lesssim \frac{1}{t^{1/2}} \left(2\epsilon e^{L\tau} + (M^2+\eta^2)M \right).\end{aligned}$$

Reproducing the same arguments, one may check that

$$d(\Phi(v),\Phi(w)) \lesssim (M^2+\eta^2) d(v,w).$$

Hence, for $M = 4\epsilon$ and η, ϵ small, the mapping $\Phi : \mathcal{F} \mapsto \mathcal{F}$ is a strict contraction, thus yielding a solution v of (6.2.2) on $(0,\infty)$. □

6.3 The case $\theta = \pi/2$: a qualitative result

Throughout this section, we consider the complex Ginzburg-Landau in the special case $\theta = \pi/2$ and $\gamma \neq \pi/2$:

$$iu_t + \Delta u + e^{i\gamma}|u|^\sigma u = 0, \quad -\pi/2 \leqslant \gamma < 0. \tag{CNLS}$$

In the case $\gamma = 0$, the equation reduces to the usual (NLS). Since the presence of $e^{i\gamma}$ has no influence in generic estimates that one may do on the nonlinearity, the standard local well-posedness techniques that apply to the (NLS) are also applicable here:

Proposition 6.3.1 (Local well-posedness in $H^1(\mathbb{R}^d)$)**.** *Fix* $0 < \sigma < 4/(d-2)^+$*. Given* $u_0 \in H^1(\mathbb{R}^d)$*, there exist* $T(u_0) > 0$ *and a unique maximal solution*

$$u \in C([0, T(u_0)); H^1(\mathbb{R}^d))$$

of (CNLS)*, which depends continuously on the initial data. If* $T(u_0) < \infty$*, then one has the blow-up alternative*

$$\lim_{t \to T(u_0)} \|u(t)\|_{H^1} = +\infty.$$

On the other hand, the presence of a non-real coefficient in the nonlinearity breaks the Hamiltonian character of the dynamical system. As such, even though one has the gauge invariance $u \mapsto e^{i\lambda}u$, the L^2 norm is not conserved: in fact,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = -\sin\gamma \|u(t)\|_{\sigma+2}^{\sigma+2} \geqslant 0.$$

Remark 6.3.1. Observe that, if the spatial domain were a bounded set Ω, then

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \geqslant -\sin\gamma |\Omega|^{-\frac{\sigma}{2}} \|u(t)\|_2^{\sigma+2}.$$

This implies that, if $u_0 \not\equiv 0$,

$$\frac{1}{\|u(t)\|_2^\sigma} \leqslant \frac{1}{\|u_0\|_2^\sigma} + \frac{\sigma \sin\gamma |\Omega|^{-\frac{\sigma}{2}} t}{2}, \ t < T(u_0).$$

Since the right-hand side is negative for large t (recall that $\sin\gamma < 0$), we conclude that $T(u_0) < \infty$. Thus all nontrivial solutions blow-up in finite time.

The behaviour of solutions over bounded domains strongly indicates that the equation over $\mathbb{R}^d$ should also present blow-up in finite time. By localizing the L^2 norm of u, one may use the ideas for bounded domains and obtain blow-up in finite or infinite time:

Theorem 6.3.2. *Fix* $\sigma < \frac{2}{d}$*. There exists* $\delta > 0$ *such that if* $u \not\equiv 0$ *is a global* H^1 *solution of* (CNLS)*, then*

$$\sup_{0 \leqslant s \leqslant t} \|\nabla u(s)\|_2 \geqslant \delta \|u(0)\|_2^{\frac{d+2}{d}} t^{\frac{2-d\sigma}{d\sigma}}, \quad t > 0. \tag{6.3.1}$$

Moreover,

$$\lim_{t \to \infty} t^{-\frac{2-d\sigma}{d\sigma}} \sup_{0 \leqslant s \leqslant t} \|\nabla u(s)\|_2 = \infty. \tag{6.3.2}$$

Remark 6.3.2. It is currently unknown if blow-up may occur in finite time. Moreover, one cannot even ensure blow-up of the L^2 norm (which would be a natural conjecture, given Remark 6.3.1)

Proof of Theorem 6.3.2. Step 1. Finite or infinite time blow-up. We fix a cut-off function $\psi(x) = \nu\theta(|x|)$, where

$$\theta(r) = \begin{cases} 1 & 0 \leqslant r \leqslant 1 \\ 2 - r & 1 \leqslant r \leqslant 2 \\ 0 & r \geqslant 2 \end{cases}$$

and $\nu \in \mathbb{R}$ is such $\|\psi\|_2 = 1$. Given $\lambda > 0$, set

$$\psi_\lambda(x) = \psi(\lambda x).$$

It follows that

$$\|\psi_\lambda\|_2 = \lambda^{-\frac{d}{2}} \quad \text{and} \quad \|\nabla\psi_\lambda\|_\infty = \nu\lambda. \tag{6.3.3}$$

We now localize the mass of u: using the duality $H^{-1} - H^1$, apply $\psi_\lambda^2 \overline{u} \in H^{-1}(\mathbb{R}^d)$ to (CNLS):

$$\frac{1}{2}\frac{d}{dt}\int |u|^2\psi_\lambda^2 = 2\,\mathrm{Im}\int \overline{u}\psi_\lambda\nabla u \cdot \nabla\psi_\lambda - \sin\gamma\int |u|^{\sigma+2}\psi_\lambda^2. \tag{6.3.4}$$

We define

$$f_\lambda(t) = \|u\psi_\lambda\|_2, \quad K_t = \|\nabla u\|_{L^\infty((0,t),L^2)}.$$

It follows from Hölder's inequality and (6.3.3) that

$$f_\lambda(t)^{\sigma+2} \leqslant \|\psi_\lambda\|_2^\sigma \int |u|^{\sigma+2}\psi_\lambda^2 = \lambda^{-\frac{d\sigma}{2}}\int |u|^{\sigma+2}\psi_\lambda^2$$

and

$$\left|\mathrm{Im}\int \overline{u}\psi_\lambda\nabla u \cdot \nabla\psi_\lambda\right| \leqslant \|\nabla\psi_\lambda\|_\infty\|\psi_\lambda u\|_2\|\nabla u\|_2 = \nu\lambda K_t f_\lambda(t).$$

These estimates together with (6.3.4) imply that, for any fixed $T > 0$,

$$f_\lambda'(t) \geqslant -2\nu\lambda K_t - \sin\gamma\lambda^{\frac{d\sigma}{2}} f_\lambda^{\sigma+1}(t) \geqslant -2\nu\lambda K_T - \sin\gamma\lambda^{\frac{d\sigma}{2}} f_\lambda^{\sigma+1}(t),\ t < T.$$

If one has

$$f_\lambda(0)^{\sigma+1} \geqslant 4(-\sin\gamma)^{-1}\lambda^{\frac{2-d\sigma}{2}}\nu K_T, \tag{6.3.5}$$

it follows that f_λ is increasing on $(0, T)$ and

$$f_\lambda'(t) \geqslant \frac{-\sin\gamma}{2}\lambda^{\frac{d\sigma}{2}} f_\lambda^{\sigma+1}(t), \quad t < T. \tag{6.3.6}$$

Integrating this inequality,

$$\frac{1}{f_\lambda(t)^\sigma} \leqslant \frac{1}{f_\lambda(0)^\sigma} + \frac{\sigma\sin\gamma\lambda^{\frac{d\sigma}{2}}t}{2}, \quad t < T.$$

Since $\sin\gamma < 0$ and f_λ is a nonnegative quantity, one must have

$$T \leqslant -\frac{2}{\sigma\sin\gamma}\lambda^{-\frac{d\sigma}{2}} f_\lambda(0)^{-\sigma}. \tag{6.3.7}$$

Suppose, by contradiction, that K_T is bounded for all $T > 0$. Since

$$\lim_{\lambda\to 0} f_\lambda(0) = \lim_{\lambda\to 0} \|\psi_\lambda u_0\|_2 = \|u_0\|_2,$$

the condition (6.3.5) is verified for any $T > 0$, as long as λ is sufficiently small. Hence (6.3.7) is valid for any $T > 0$, which is absurd. Hence K_T is unbounded. Since K_T is nondecreasing, we conclude that

$$\lim_{T\to\infty} K_T = \infty. \tag{6.3.8}$$

Step 2. Proof of estimate (6.3.1). Since, by Dominated Convergence, the function $\lambda \mapsto f_\lambda(0)$ i continuous and

$$\lim_{\lambda\to 0} f_\lambda(0) = \|u_0\|_2, \quad \lim_{\lambda\to\infty} f_\lambda(0) = 0,$$

there exists $\lambda_0 > 0$ such that

$$f_{\lambda_0}(0) = \frac{1}{2}\|u_0\|_2. \tag{6.3.9}$$

On the other hand, define $\lambda(T)$ so that

$$f_{\lambda_0}(0)^{\sigma+1} = 4(-\sin\gamma)^{-1}\lambda(T)^{\frac{2-d\sigma}{2}}\nu K_T.$$

From (6.3.8), if T is large enough,

$$\lambda(T) \leqslant \lambda_0.$$

Since $f_\lambda(0)$ is a nonincreasing function of λ,

$$f_{\lambda(T)}(0)^{\sigma+1} \geqslant f_{\lambda_0}(0)^{\sigma+1} = 4(-\sin\gamma)^{-1}\lambda(T)^{\frac{2-d\sigma}{2}}\nu K_T,$$

which means that (6.3.5) is valid for $\lambda(T)$. Therefore, by (6.3.7),

$$T \leqslant \frac{2}{-\sigma\sin\gamma}\lambda(T)^{-\frac{d\sigma}{2}} f_{\lambda(T)}(0)^{-\sigma} \leqslant \frac{2}{-\sigma\sin\gamma}\lambda(T)^{-\frac{d\sigma}{2}} f_{\lambda_0}(0)^{-\sigma}.$$

Using the definition of $\lambda(T)$, one has

$$\begin{aligned} T^{\frac{2-d\sigma}{d\sigma}} &\leqslant \frac{4}{-\sin\gamma}\Big(\frac{2}{-\sigma\sin\gamma} f_{\lambda_0}(0)^{-\sigma}\Big)^{\frac{2-d\sigma}{d\sigma}} f_{\lambda_0}(0)^{-(\sigma+1)}\nu K_T \\ &= 2^{\frac{2+d\sigma}{d\sigma}}(-\sin\gamma)^{-\frac{2}{d\sigma}}\sigma^{-\frac{2-d\sigma}{d\sigma}} f_{\lambda_0}(0)^{-\frac{d+2}{d}}\nu K_T. \end{aligned} \tag{6.3.10}$$

Inequality (6.3.1) now follows from (6.3.10) and (6.3.9).

Step 3. Proof of (6.3.2). Given $T > 0$, we define $\mu(T)$ so that

$$f_{\mu(T)}(0)^{\sigma+1} = 4(-\sin\gamma)^{-1}\mu(T)^{\frac{2-d\sigma}{2}}\nu K_T.$$

It follows easily from the monotonicity of the mappings

$$\mu \mapsto f_\mu(0), \quad T \mapsto K_T$$

that the mapping $T \mapsto \mu(T)$ is nonincreasing. From Step 1, we have

$$4(-\sin\gamma)^{-1}\mu(T)^{\frac{2-d\sigma}{2}} = \frac{f_{\mu(T)}(0)^{\sigma+1}}{K_T} \leqslant \frac{\|u_0\|_2^{\sigma+1}}{\nu K_T} \to 0,\ T \to \infty,$$

and so $\mu(T) \to 0$ as $T \to \infty$. As a consequence,

$$\lim_{T\to\infty} \mu(T)^{\frac{2-d\sigma}{2}} K_T = \lim_{T\to\infty} \frac{-\sin\gamma f_{\mu(T)}(0)^{\sigma+1}}{4\nu} = \frac{-\sin\gamma \|u_0\|_2^{\sigma+1}}{4\nu}$$

and so there exists $\eta > 0$ such that

$$\mu(T)^{\frac{2-d\sigma}{2}} K_T > \eta^{\frac{2-d\sigma}{d\sigma}}, \quad T \geqslant 2. \tag{6.3.11}$$

The definition of $\mu(T)$ implies that (6.3.5) is true for $\lambda = \mu(T)$. By (6.3.6),

$$f'_{\mu(T)} \geqslant \frac{-\sin\gamma}{2}\mu(T)^{\frac{d\sigma}{2}} f_{\mu(T)}^{\sigma+1}, \quad 0 < t < T.$$

Integrating the above differential inequality on $(T/2, T)$ and using (6.3.11), we obtain, for $T \geqslant 2$,

$$f_{\mu(T)}(T/2)^{-\sigma} - f_{\mu(T)}(T)^{-\sigma} \geqslant \frac{-\sin\gamma\sigma}{4}\mu(T)^{\frac{d\sigma}{2}} T \geqslant \eta K_T^{-\frac{d\sigma}{2-d\sigma}} T.$$

Since $\mu(T) \leqslant \mu(T/2)$,

$$f_{\mu(T/2)}(T/2)^{-\sigma} - f_{\mu(T)}(T)^{-\sigma} \geqslant \eta K_T^{-\frac{d\sigma}{2-d\sigma}} T, \quad T \geqslant 2. \tag{6.3.12}$$

One easily observes that

$$f_{\mu(s)}(s) \leqslant f_{\mu(t)}(s) \leqslant f_{\mu(t)}(t), \quad s \leqslant t.$$

Therefore the mapping $t \mapsto f_{\mu(t)}(t)^{-\sigma}$ is nonincreasing, so it has a limit as $t \to \infty$. Taking the limit $T \to \infty$ in (6.3.12), we deduce that

$$\lim_{T\to\infty} K_T^{-\frac{d\sigma}{2-d\sigma}} T = 0$$

which is equivalent to (6.3.2). □

Under some restrictions on γ, one may strengthen Theorem 6.3.2: in fact, using the next lemma, one can replace $\sup_{0\leqslant s\leqslant t} \|\nabla u(s)\|_2$ by $\|\nabla u(t)\|_2$. Notice that (6.3.13) is identical to condition (2.2) in [61].

Lemma 6.3.3. *Suppose that γ is such that*

$$-\sin\gamma \geqslant \frac{\sigma}{\sigma+2}. \tag{6.3.13}$$

Then, for any H^1 solution u of (CNLS)*, the mapping $t \mapsto \|\nabla u(t)\|_2$ is nondecreasing.*

Proof. Formally, multiplying (CNLS) by $\Delta\bar{u}$, integrating over $\mathbb{R}^d$ and taking the imaginary part,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_2^2 &= -\operatorname{Im}\Big(e^{i\gamma}\int \nabla(|u|^\sigma u)\cdot\nabla\overline{u}\Big) \\
&= -\operatorname{Im}\left(e^{i\gamma}\frac{\sigma+2}{2}\int |u|^\sigma|\nabla u|^2 + \frac{\sigma}{2}\int |u|^{\sigma-2}u^2(\nabla\overline{u})^2\right) \\
&= -\sin\gamma\frac{\sigma+2}{2}\int |u|^\sigma|\nabla u|^2 + \frac{\sigma}{2}\operatorname{Im} e^{i\gamma}\int |u|^{\sigma-2}u^2(\nabla\overline{u})^2 \\
&\geqslant \left(-\sin\gamma\frac{\sigma+2}{2} - \frac{\sigma}{2}\right)\int |u|^\sigma|\nabla u|^2 \geqslant 0.
\end{aligned}$$

These computations are valid if u is, for example, an H^2 solution. The general case follows by continuous dependence. □

Remark 6.3.3. As it is well-known, in the (NLS) case, one has global well-posedness and scattering for oscillating data. Since the proof relies only on the properties of the Schrödinger group and not on the specific coefficient of the nonlinearity, such a result is also valid in the context of the (CNLS). Thus we see that the exponent $\sigma = 2/d$ is critical in terms of qualitative behaviours. Let us give an heuristic argument for the criticallity of $\sigma = 2/d$: if one looks for solutions of (CNLS) of the form

$$u(t,x) = \rho(t)e^{i\frac{|x|^2}{4(t+t_0)}},$$

where $t_0 > 0$ is given, then ρ must satisfy

$$\rho' = -\frac{d}{2(t+t_0)}\rho - ie^{i\gamma}|\rho|^\sigma\rho.$$

Setting $z = (t+t_0)^{\frac{d}{2}}\rho$, we get to the equation

$$z' = -ie^{i\gamma}(t+t_0)^{-\frac{d\sigma}{2}}|z|^\sigma z.$$

Multiplying the equation by $\overline{z}$ and taking the real part, one easily gets to

$$\frac{1}{\sigma|z(t)|^\sigma} = \frac{1}{\sigma|z(0)|^\sigma} + \sin\gamma\int_0^t \frac{ds}{(s+t_0)^{\frac{d\sigma}{2}}}. \tag{6.3.14}$$

If $\sigma > \frac{2}{d}$, then the integral on the right-hand side of (6.3.14) is convergent, and we see that if $|z(0)|$ is sufficiently small so that

$$\frac{1}{\sigma|z(0)|^\sigma} \geqslant -\sin\gamma \int_0^\infty \frac{ds}{(s+t_0)^{\frac{d\sigma}{2}}},$$

then the solution is global; and if $|z(0)|$ is larger, then the solution blows up in finite time. On the other hand, if $\sigma \leqslant \frac{2}{d}$, then the integral on the right-hand side of (6.3.14) is divergent. Therefore, for every $z(0)$, the solution blows up at the finite time T given by

$$\frac{1}{\sigma|z(0)|^\sigma} = -\sin\gamma \int_0^T \frac{ds}{(s+t_0)^{\frac{d\sigma}{2}}}.$$

Chapter 7

An abstract evolution problem with a nonlocal nonlinearity

7.1 Introduction

Let X be a complex Hilbert space (with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$). If one defines $(\cdot,\cdot) = \mathrm{Re}\langle\cdot,\cdot\rangle$, then X is a real Hilbert space with this scalar product. Let A be a negative, self-adjoint linear operator on X with dense domain $D(A)$. It follows that A is the infinitesimal generator of a contraction semigroup $\{e^{tA}\}_{t\geqslant 0}$. Since A is self-adjoint, iA is skew-adjoint and we note by $\{e^{itA}\}_{t\in\mathbb{R}}$ the isometry group generated by iA. Let

$$-\frac{\pi}{2} < \theta, \gamma < \frac{\pi}{2}.$$

Then $e^{i\theta}A$ is the generator of the contraction semigroup $\{S_\theta(t)\}_{t\geqslant 0}$, with

$$S_\theta(t) = e^{it\sin\theta A}e^{t\cos\theta A}.$$

Fix $\sigma > 0$. Given any $u_0 \in X$, consider the initial value problem

$$\begin{cases} u' = e^{i\theta}Au + e^{i\gamma}\|u\|^\sigma u, \\ u(0) = u_0. \end{cases} \tag{7.1.1}$$

in the equivalent form

$$u(t) = S_\theta(t)u_0 + e^{i\gamma}\int_0^t S_\theta(t-s)\|u(s)\|^\sigma u(s)ds \tag{7.1.2}$$

It is easy to see that this problem is locally well-posed for any $u_0 \in X$, giving a unique maximal solution $u \in C([0, T_{max}(u_0)); X)$. The maximal time of existence $T_{max}(u_0)$ for the corresponding solution u is finite if and only if $\|u(t)\| \to \infty$ as $t \to T_{max}(u_0)$.

The above problem can be seen as an attempt to better understanding the blowup mechanics related with power nonlinearities, present in equations such as the nonlinear heat equation and, more generally, the complex Ginzburg-Landau equation. The most surprising fact about this evolution problem is that it has an explicit solution. As a consequence, one is able to determine precisely the set $\mathcal{G}$ of initial data for which the corresponding solution is global and study geometrical properties for this set, such as convexity and connectedness (see Section 7.2). Using the explicit formula, one can see that (at least for $\sigma \geqslant 1$), the set $\mathcal{G}$ is the unit ball for a special norm. In Section 7.3, assuming that A is invertible, we proceed to study the relationship between the space defined by this norm and some interpolation spaces that one may construct using X and $D(A^{-1})$. Moreover, we prove local well-posedness for these new spaces (Section 7.4). Finally, in Section 7.5, we discuss some applications to nonlinear heat equations and obtain, in an easy way, global well-posedness for small data.

7.2 Explicit solution and blow-up characterization

Definition 7.2.1. *For any $u \in X$, if*

$$\sigma \cos\gamma \int_0^\infty \|S_\theta(s)u\|^\sigma ds > 1$$

define $T(u)$ as the only solution of the equation

$$\sigma \cos\gamma \int_0^{T(u)} \|S_\theta(s)u\|^\sigma ds = 1$$

(the uniqueness follows from the classical result of [48]). Otherwise, set $T(u) = \infty$.

Proposition 7.2.2. *For any $u_0 \in X$,*

$$u(t) = \frac{S_\theta(t)u_0}{\left[1 - \sigma\cos\gamma \int_0^t \|S_\theta(s)u_0\|^\sigma ds\right]^{\frac{e^{i\gamma}}{\sigma\cos\gamma}}}, \quad 0 \leqslant t < T(u_0) \tag{7.2.1}$$

is the unique maximal solution of problem (7.1.1) *in the class*

$$C([0, T(u_0)), X) \cap C^1((0, T(u_0)), D(A)).$$

Consequently, the set of initial data $u_0 \in X$ that give rise to global solutions is precisely

$$\mathcal{G} := \{u_0 \in X : T_{max}(u_0) = \infty\} = \left\{u_0 \in X : \sigma\cos\gamma \int_0^\infty \|S_\theta(t)u_0\|^\sigma dt \leqslant 1\right\}.$$

Proof. It can easily be seen that $u \in C([0, T(u_0)), X) \cap C^1((0, T(u_0)), D(A))$. For any $0 < t < T(u_0)$,

$$\begin{aligned} u'(t) &= \frac{e^{i\theta} A S_\theta(t)u_0}{\left[1 - \sigma\cos\gamma \int_0^t \|S_\theta(s)u_0\|^\sigma ds\right]^{\frac{e^{i\gamma}}{\sigma\cos\gamma}}} + e^{i\gamma} \frac{S_\theta(t)u_0 \|S_\theta(t)u_0\|^\sigma}{\left[1 - \sigma\cos\gamma \int_0^t \|S_\theta(s)u_0\|^\sigma ds\right]^{\frac{e^{i\gamma}}{\sigma\cos\gamma}+1}} \\ &= e^{i\theta} Au(t) + e^{i\gamma}\|u(t)\|^\sigma u(t). \end{aligned}$$

Since $u(0) = u_0$, then $u(t)$ is the (only) solution of (7.1.1) over the interval $[0, T(u_0))$ in the class $C([0, T(u_0)), X) \cap C^1((0, T(u_0)), D(A))$. Moreover, since $\|u(t)\| \to \infty$ as $t \to T(u_0)$, the blow-up alternative implies that $T(u_0) = T_{max}(u_0)$. $\square$

REMARK 7.2.1. Since $S_\theta(t) = e^{it\sin\theta A} e^{t\cos\theta A}$, a change of variables yields

$$\mathcal{G} = \left\{ u_0 \in X : \int_0^\infty \|e^{tA} u_0\|^\sigma \leqslant \frac{\cos\theta}{\sigma\cos\gamma} \right\}.$$

REMARK 7.2.2. If the spectrum of A is bounded above by $-m < 0$, then

$$\int_0^\infty \|e^{tA} u_0\|^\sigma dt \leqslant \|u_0\|^\sigma \int_0^\infty e^{-\sigma m t} dt = \frac{1}{\sigma m} \|u_0\|^\sigma.$$

This means that there exists $r > 0$ such that $B_r(0) \subset \mathcal{G}$. Also, if the spectrum of A is bounded below by a constant $-M < 0$,

$$\int_0^\infty \|e^{tA} u_0\|^\sigma dt \geqslant \frac{1}{\sigma M} \|u_0\|^\sigma,$$

and so there exists $R > 0$ such that $\mathcal{G} \subset B_R(0)$.

REMARK 7.2.3. The above result can be also extended for the more general abstract equation

$$\begin{cases} u'(t) = e^{i\theta} A u(t) + e^{i\gamma} F(t, u) u(t), \\ u(0) = u_0. \end{cases}$$

where $F \in C([0, \infty) \times X; \mathbb{R})$ satisfies, for a fixed $\sigma > 0$,

$$F(t, \lambda x) = |\lambda|^\sigma F(t, x), \quad \forall \lambda \in \mathbb{C} \quad \forall (t, x) \in [0, \infty).$$

Proposition 7.2.3. *Fixed $\sigma > 0$, one has the following:*

1. *The set $\mathcal{G}$ is star-shaped and symmetric;*
2. *If $\sigma \geqslant 1$, $\mathcal{G}$ is convex;*
3. *If $\sigma < 1$, let λ be an eigenvalue of A. Then there exists $M(\lambda) > 0$ such that, if the diameter of the spectrum of A is larger than $M(\lambda)$, then $\mathcal{G}$ is not convex.*

Proof. The first statement is obvious, since, if $u_0 \in \mathcal{G}$, $ku_0 \in \mathcal{G}$, for any $k \in [-1, 1]$. The second is a direct consequence of the convexity of the map $u \mapsto \|u\|^\sigma$.

We now prove the last assertion. Let ϕ_1, ϕ_2 be two eigenvectors of A, normalized in X, associated, respectively, with the eigenvalues $-\lambda_1, -\lambda_2 < 0$.

It is easy to see that, for $a_i = (\lambda_i \cos\theta / \cos\gamma)^{1/\sigma}$, $i = 1, 2$,

$$\int_0^\infty \|e^{tA} a_i \phi_i\|^\sigma dt = \frac{\cos\theta}{\sigma\cos\gamma}, \; i = 1, 2,$$

so that $a_i\phi_i \in \mathcal{G},\ i = 1, 2$. We claim that, for any $\eta \in (0, 1)$, when λ_1 is fixed and λ_2 is sufficiently large, one has

$$\int_0^\infty \|S_\theta(t)\,(\eta a_1\phi_1 + (1-\eta)a_2\phi_2)\,\|^\sigma dt > \eta \int_0^\infty \|S_\theta(t)a_1\phi_1\|^\sigma dt + (1-\eta)\int_0^\infty \|S_\theta(t)a_2\phi_2\|^\sigma dt = \frac{\cos\theta}{\sigma\cos\gamma},$$

which proves that $\mathcal{G}$ is not convex.

Fix any $\eta \in (0, 1)$ and let

$$f(x, y) = \left(\eta^2x^2 + (1-\eta)^2y^2\right)^{\sigma/2} - \eta x^\sigma - (1-\eta)y^\sigma, x, y \in \mathbb{R}^+.$$

When $y/x \to 0$,

$$\frac{f(x, y)}{x^\sigma} \to \eta^\sigma - \eta > 0,$$

which means that there exists $D > 0$ such that

$$f(x, y) \geqslant \frac{\eta^\sigma - \eta}{2}x^\sigma > 0,\ \forall\ y \leqslant Dx \tag{7.2.2}$$

One can easily check that, for $t \geqslant T_{\lambda_2} = \frac{\log(a_2/Da_1)}{\lambda_2-\lambda_1}$,

$$Da_1e^{-\lambda_1 t} \geqslant a_2e^{-\lambda_2 t}.$$

Notice that $T_{\lambda_2} \to 0$ as $\lambda_2 \to \infty$. Then, using (7.2.2), one has

$$\left(\eta^2a_1^2e^{-2\lambda_1 t} + (1-\eta)^2a_2^2e^{-2\lambda_2 t}\right)^{\sigma/2} - \eta a_1^\sigma e^{-\sigma\lambda_1 t} - (1-\eta)a_2^\sigma e^{-\sigma\lambda_2 t} \geqslant \frac{\eta^\sigma - \eta}{2}a_1^\sigma e^{-\sigma\lambda_1 t}$$

for any $t > T_{\lambda_2}$. We conclude that

$$\begin{aligned}
&\int_0^\infty \|S_\theta(t)\,(\eta a_1\phi_1 + (1-\eta)a_2\phi_2)\,\|^\sigma dt - \eta\int_0^\infty \|S_\theta(t)a_1\phi_1\|^\sigma dt \\
&\qquad\qquad - (1-\eta)\int_0^\infty \|S_\theta(t)a_2\phi_2\|^\sigma dt \\
&= \int_0^\infty \left(\eta^2a_1^2e^{-2\lambda_1 t} + (1-\eta)^2a_2^2e^{-2\lambda_2 t}\right)^{\sigma/2} - \eta a_1^\sigma e^{-\sigma\lambda_1 t} - (1-\eta)a_2^\sigma e^{-\sigma\lambda_2 t}dt \\
&> \int_0^{T_{\lambda_2}} \left(\eta^2a_1^2e^{-2\lambda_1 t} + (1-\eta)^2a_2^2e^{-2\lambda_2 t}\right)^{\sigma/2} - \eta a_1^\sigma e^{-\sigma\lambda_1 t} - (1-\eta)a_2^\sigma e^{-\sigma\lambda_2 t}dt \\
&+ \int_{T_{\lambda_2}}^\infty \frac{\eta^\sigma - \eta}{2}a_1^\sigma e^{-\sigma\lambda_1 t}dt \geqslant -\int_0^{T_{\lambda_2}} \eta a_1^\sigma e^{-\sigma\lambda_1 t} + \frac{\eta^\sigma - \eta}{2}\frac{e^{-\sigma\lambda_1 T_{\lambda_2}}}{\sigma\lambda_1} > 0
\end{aligned}$$

for sufficiently large λ_2. □

If $Ax = kx$ for some $k < 0$, then the set $\mathcal{G}$ is just a ball in X and therefore is convex, regardless of σ. One may ask if, when the diameter of the spectrum of A is small, the set $\mathcal{G}$ remains convex for any $\sigma > 0$.

Proposition 7.2.4. *Fix $\sigma > 0$. Suppose that X has a basis $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenvectors of A and that the first eigenvalue λ_1 of $-A$ is strictly positive. If the diameter of the spectrum of A is sufficiently small, then $\mathcal{G}$ is convex.*

Proof. Suppose that the spectrum of A is bounded from below by a constant $-M < 0$. It follows from Remark 2.2 that there exist constants $r, R > 0$ such that $B_r(0) \subset \mathcal{G} \subset B_R(0)$. Let $x, y \in \mathcal{G}$ be such that the segment between them does not intersect $B_{r/2}(0)$ and consider $V = span\{x, y\}$. If $\dim V = 0, 1$, then it is clear that the segment between x and y is in $\mathcal{G}$. Suppose that $\dim V = 2$ and also that $\|x\| \geqslant \|y\|$. Let $x' \in V$ be an orthogonal vector to x with norm $\|x\|$ and define

$$f(a,b) = \left(\int_0^\infty e^{-\sigma\lambda_1 t}\|ax + bx'\|^\sigma dt\right)^{2/\sigma}, \; g(a,b) = \left(\int_0^\infty \|e^{tA}(ax + bx')\|^\sigma dt\right)^{2/\sigma}, \; a, b \in \mathbb{R}$$

Observe that f is a strictly convex function. Writing $x = \sum_k a_k\phi_k$ and $x' = \sum_k b_k\phi_k$, then

$$g(a,b) = \left(\int_0^\infty \left(\sum_k e^{-2\lambda_k t}(aa_k + bb_k)^2\right)^{\sigma/2} dt\right)^{2/\sigma}$$

and

$$f(a,b) = \left(\int_0^\infty e^{-\sigma\lambda_k t} \left(\sum_k (aa_k + bb_k)^2\right)^{\sigma/2} dt\right)^{2/\sigma}$$

One has

$$\left|\left(\sum_k (aa_k + bb_k)^2 e^{-2\lambda_k t}\right)^{\sigma/2} - e^{-\sigma_1\lambda t}\left(\sum_k (aa_k + bb_k)^2\right)^{\sigma/2}\right|$$
$$\leqslant \left(\sum_k (aa_k + bb_k)^2\right)^{\sigma/2}\left(e^{-\sigma\lambda_1 t} - e^{-\sigma\max_k\lambda_k t}\right),$$

which means that

$$g(a,b) \to f(a,b) \text{ as } \max_k \lambda_k - \lambda_1 \to 0,$$

uniformly over $C = \{(a,b) \in \mathbb{R}^2 : r^2/4R^2 \leqslant a^2 + b^2 \leqslant 4\}$ and on x and y (since $x, y \in B_R(0)$). Also, one easily checks that the same holds true for their first and second derivatives. Therefore, for $\max_k \lambda_k - \lambda_1 < \epsilon$ sufficiently small, the Hessian matrix of g is positive definite on C (since it converges to a strictly positive definite matrix - the Hessian matrix of f - uniformly in that set). Hence g is convex in any convex subset of C. Recalling that $\|y\| \leqslant \|x\|$ and $\|x'\| = \|x\|$, there exists $(a_0, b_0) \in \mathbb{R}^2$ such that $y = a_0x + b_0x'$ and $a_0^2 + b_0^2 \leqslant 1$. From the hypothesis that the segment between x and y does not intersect $B_{r/2}(0)$, it follows that the segment between $(1,0)$ and (a_0, b_0) is inside C. Since g is convex in any convex subset of C, we have

$$g(\eta(1,0) + (1-\eta)(a_0,b_0)) \leqslant \eta g(1,0) + (1-\eta)g(a_0,b_0), \; \forall\eta \in (0,1),$$

which implies that the segment between x and y is in $\mathcal{G}$. Since the smallness of ϵ does not depend on x and y, we obtain that, if $\max_k \lambda_k - \lambda_1 < \epsilon$, then, for all x, y such that the segment between them does not intersect $B_{r/2}(0)$, that same segment is in $\mathcal{G}$. Since $B_r(0) \subset \mathcal{G}$, one may easily observe that the segment between any two points $x, y \in \mathcal{G}$ is inside $\mathcal{G}$ (regardless of whether the segment intersects $B_{r/2}(0)$). □

REMARK 7.2.4. The size of the spectrum in the previous result depends on λ_1. This is not unexpected, since, in Proposition 7.2.3, M also depended on a fixed eigenvalue of A (which we might take as λ_1).

7.3 The functional space defined by the blow-up norm

Throughout this section, we shall suppose that, for some $\delta > 0$,

$$(-Au, u) \geqslant \delta \|u\|^2.$$

This implies that A is invertible. Moreover, we assume that X has a basis $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenvectors of A. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the corresponding set of eigenvalues. Thus one may define the spaces $D(A^{-1/\beta})$, for any $\beta \geqslant 1$, as the space of formal linear combinations $u = \sum_{k>0} a_k\phi_k$ such that

$$\|u\|^2_{D(A^{-1/\beta})} := \sum_{k>0} \frac{a_k^2}{\lambda_k^{2/\beta}} < \infty.$$

The fact that the set $\mathcal{G}$ is the unit ball for a specific norm is quite interesting. One may wonder if this norm is equivalent to some other already known norm. More precisely, if one defines, for any $\sigma \geqslant 1$ and $u_0 \in D(A)$,

$$\|u_0\|_{\dot{E}_\sigma} = \left(\int_0^\infty \|e^{tA}u_0\|^\sigma dt \right)^{1/\sigma}$$

and take the completion of $D(A)$ for this norm,

$$\dot{E}_\sigma := \overline{D(A)}^{\|\cdot\|_{\dot{E}_\sigma}},$$

does this space coincide with an interpolation space of X and $D(A^{-1})$? Since any interpolation space will lie in $D(A^{-1})$, one should consider the restricted space

$$E_\sigma = D(A^{-1}) \cap \dot{E}_\sigma.$$

REMARK 7.3.1. If the spectrum of A is bounded, then, by Remark 7.2.2, $E_\sigma = X$, for all $\sigma \geqslant 1$. Therefore, we shall suppose that the spectrum of A is unbounded throughout this section.

Lemma 7.3.1. *If $\sigma > \beta \geqslant 1$, then $X \hookrightarrow E_\sigma \hookrightarrow E_\beta$. Moreover, $E_1 \simeq D(A^{-1})$.*

Proof. The injection $X \hookrightarrow E_\sigma$ is a direct consequence of the exponential decay of the semigroup. Fix $u \in D(A)$. Recall the decay estimate of the semigroup $\{e^{tA}\}_{t\geqslant 0}$,

$$\|e^{tA}u\| \leqslant (1/\sqrt{2}t)\|u\|_{D(A^{-1})}.$$

Then, given $u \in X$,

$$\begin{aligned}
\|u\|_{\dot{E}_\beta}^\beta &\lesssim \int_0^{\frac{\|u\|_{D(A^{-1})}}{\sqrt{2}}} \|e^{tA}u\|^\beta dt + \int_{\frac{\|u\|_{D(A^{-1})}}{\sqrt{2}}}^\infty \|e^{tA}u\|^\beta dt \\
&\lesssim \|u\|_{D(A^{-1})}^{\frac{\sigma-\beta}{\sigma}} \left(\int_0^{\frac{\|u\|_{D(A^{-1})}}{\sqrt{2}}} \|e^{tA}u\|^\sigma dt \right)^{\frac{\beta}{\sigma}} + \int_{\frac{\|u\|_{D(A^{-1})}}{\sqrt{2}}}^\infty \|e^{tA}u\|^\sigma dt \\
&\lesssim \|u\|_{D(A^{-1})} + \|u\|_{\dot{E}_\sigma}^\sigma.
\end{aligned}$$

Hence, given any $\lambda > 0$,

$$\|\lambda u\|_{\dot{E}_\beta}^\beta \lesssim \|\lambda u\|_{D(A^{-1})} + \|\lambda u\|_{\dot{E}_\sigma}^\sigma, \text{ i.e., } \|u\|_{\dot{E}_\beta}^\beta \lesssim \lambda^{1-\beta}\|u\|_{D(A^{-1})} + \lambda^{\sigma-\beta}\|u\|_{\dot{E}_\sigma}^\sigma.$$

When $\beta = 1$, the limit $\lambda \to 0$ gives $D(A^{-1}) \hookrightarrow \dot{E}_1$ and so $E_1 \simeq D(A^{-1})$. When $\beta > 1$, the optimization in λ gives

$$\|u\|_{\dot{E}_\beta} \lesssim \|u\|_{D(A^{-1})}^{\frac{\sigma-\beta}{\beta(\sigma-1)}} \|u\|_{\dot{E}_\sigma}^{\frac{\sigma(\beta-1)}{\beta(\sigma-1)}} \lesssim \|u\|_{E_\sigma}.$$

□

Since the complex interpolation of X and $D(A^{-1})$ is $D(A^{-1/\beta})$, let us compute the norm of each eigenvector ϕ_k in E_σ and $D(A^{-1/\beta})$:

$$\|\phi_k\|_{E_\sigma} = \left(\int_0^\infty e^{-\sigma\lambda_k t} dt\right)^{1/\sigma} = \frac{1}{(\sigma\lambda_k)^{1/\sigma}}, \quad \|\phi_k\|_{D(A^{-1/\beta})} = \frac{1}{\lambda_k^{1/\beta}}.$$

It becomes clear that, if $\sigma > \beta$, $D(A^{-1/\beta}) \not\hookrightarrow E_\sigma$ and, if $\sigma < \beta$, $E_\sigma \not\hookrightarrow D(A^{-1/\beta})$. Thus the only possibility for an identification is to consider $\beta = \sigma$. A simple calculation proves that, for $\sigma = 2$, the spaces are, in fact, identical: since $u(t) = S(t)u_0$ satisfies $u_t = Au$, then taking the scalar product in $D(A^{-1/2})$ with u, one obtains

$$\frac{1}{2}\frac{d}{dt}\|u\|_{D(A^{-1/2})}^2 = -\|u\|^2.$$

Integrating in $(0, \infty)$ gives the equality $\|u_0\|_{D(A^{-1/2})} = \sqrt{2}\|u_0\|_{E_2}$.

Considering the above, one may hope that, for any σ, $D(A^{-1/\sigma}) \simeq E_\sigma$. However, we have the following result:

Proposition 7.3.2. *Fixed $\sigma \geqslant 1$,*

1. *If $\sigma \geqslant 2$, $D(A^{-1/\sigma}) \hookrightarrow E_\sigma$;*

2. *If $\sigma \leqslant 2$, $E_\sigma \hookrightarrow D(A^{-1/\sigma})$;*

3. *For any $\beta > \sigma$, $D(A^{-1/\beta}) \hookrightarrow E_\sigma$;*

4. *If $\sigma > 2$ is even, $E_\sigma \not\approx D(A^{-1/\sigma})$.*

REMARK 7.3.2. The hypothesis in the last statement is needed for technical reasons. We conjecture that $E_\sigma \not\approx D(A^{-1/\sigma})$, at least for $\sigma > 2$.

Proof. We begin with the first two statements. Fix $u_0 \in D(A)$ and write $u_0 = \sum a_k\phi_k$. Then

$$\|u_0\|^\sigma_{\dot{E}_\sigma} = \int_0^\infty \left(\sum_{k\geqslant 1} a_k^2 e^{-2\lambda_k t}\right)^{\sigma/2} dt \quad \text{and} \quad \|u_0\|^\sigma_{D(A^{-1/\sigma})} \approx \left(\sum_{k\geqslant 1} \frac{a_k^2}{\lambda_k^{2/\sigma}}\right)^{\sigma/2}$$

Define $b_k = a_k^2/\lambda_k^{2/\sigma}$ and suppose, without loss of generality, that $\sum b_k = 1$. If $\sigma \geqslant 2$, the real function $x \mapsto x^{\sigma/2}$ is convex, and so

$$\begin{aligned}\|u_0\|^\sigma_{\dot{E}_\sigma} =& \int_0^\infty \left(\sum_{k\geqslant 1} b_k\lambda_k^{2/\sigma} e^{-2\lambda_k t}\right)^{\sigma/2} dt \leqslant \sum_{k\geqslant 1} b_k \int_0^\infty \lambda_k e^{-\sigma\lambda_k t} dt = \frac{1}{\sigma}\sum_{k\geqslant 1} b_k \\ \leqslant& \frac{1}{\sigma}\|u_0\|^\sigma_{D(A^{-1/\sigma})}.\end{aligned}$$

If $\sigma \leqslant 2$, then the function $x \mapsto x^{\sigma/2}$ is concave and the inequality is reversed. The result follows by density.

The third assertion is a consequence of the decay properties of the semigroup. Since, for any $u_0 \in D(A)$, $\|e^{tA}u_0\| \leqslant e^{-\delta t}\|u_0\|$ and $\|e^{tA}u_0\| \leqslant (1/\sqrt{2}t)\|u_0\|_{D(A^{-1})}$, by interpolation, one has $\|e^{tA}u_0\| \leqslant (1/\sqrt{2}t)^{1/\beta}\|u_0\|_{D(A^{-1/\beta})}$, for all $\beta \geqslant 1$. Then, since $\beta > \sigma$,

$$\begin{aligned}\|u_0\|^\sigma_{\dot{E}_\sigma} =& \int_0^\infty \|e^{tA}u_0\|^\sigma \leqslant \int_0^1 \frac{1}{(\sqrt{2}t)^{\sigma/\beta}}\|u_0\|^\sigma_{D(A^{-1/\beta})} + \int_1^\infty e^{-\delta(t-1)}\|u_0\|^\sigma_{D(A^{-1/\beta})} \\ \leqslant& C\|u_0\|^\sigma_{D(A^{-1/\beta})}.\end{aligned}$$

We now prove the last statement, only in the case $\sigma = 4$, by building a suitable counterexample. Choose a sequence $\{\lambda_m\}_{m\in\mathbb{N}}$ of eigenvalues of A such that, for any $m \geqslant 1$,

$$\sum_{j=1}^{m-1} \frac{\lambda_m^{1/2}\lambda_j^{1/2}}{\lambda_m + \lambda_j} < 1.$$

Let ψ_m be a normalized eigenvector with eigenvalue λ_m and define $u^N = \sum_{m=1}^N \lambda_m^{1/4}\psi_m$. Then $\|u^N\|^4_{D(A^{-1/4})} = N^2$ and

$$\|u^N\|^4_{\dot{E}_4} = \int_0^\infty \left(\sum_{m=1}^N \lambda_m^{1/2} e^{-2\lambda_m t}\right)^2 dt = \sum_{m=1}^N\sum_{j=1}^N \frac{\lambda_m^{1/2}\lambda_j^{1/2}}{2(\lambda_m + \lambda_j)}$$

$$= \frac{N}{4} + \sum_{m=1}^{N} \sum_{j=1}^{m-1} \frac{\lambda_m^{1/2} \lambda_j^{1/2}}{\lambda_m + \lambda_j} \leqslant \frac{5N}{4}.$$

If there exists an universal constant C such that $\|u^N\|_{D(A^{-1/4})} \leqslant C\|u^N\|_{E_4}$, then $N^2 \leqslant \frac{5C^4 N}{4}$, for all $N \in \mathbb{N}$, which is absurd. $\square$

Remark 7.3.3. It is natural to define

$$E_\infty = \left\{ u_0 \in D(A^{-1}) : \sup_{t \in (0,\infty)} \|e^{tA} u_0\|^\sigma < \infty \right\}.$$

It is obvious that $E_\infty = X$. Since $D(A^{-1/\sigma})$ is a complex interpolation space between X and $D(A^{-1})$, one concludes that, in general, E_σ is not the complex interpolation between E_∞ and E_2.

Another way of interpolating function spaces in the real interpolation method (see [7]): given two Banach spaces A_0 and A_1 for which the sum and the intersection is well-defined, set

$$\|u\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty t^{-\theta} \left(\inf_{u=u_0+u_1} \{ \|u_0\|_{A_0} + t\|u_1\|_{A_1} \} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \; u \in A_0 + A_1$$

and define

$$(A_0, A_1)_{\theta,q} = \left\{ u \in A_0 + A_1 : \|u\|_{(A_0,A_1)_{\theta,q}} < \infty \right\}$$

Proposition 7.3.3. *Given $1 < \sigma < \infty$, we have*

$$E_\sigma \simeq (D(A^{-1}), X)_{1-\frac{1}{\sigma},\sigma} = (E_1, E_\infty)_{1-\frac{1}{\sigma},\sigma}.$$

Proof. From [10, Theorem 3.5.3], we have

$$\|f\|_{(D(A),X)_{1-\frac{1}{\sigma},\sigma}} \cong \|f\| + \left(\int_0^\infty \|S(t)Af\|^\sigma \right)^{\frac{1}{\sigma}} = \|Af\|_{E_\sigma}.$$

Since $A : D(A) \to X$ and $A : X \to D(A^{-1})$ are isomorphisms, for $u = Af$, one has

$$\|u\|_{E_\sigma} \cong \|f\|_{(D(A),X)_{1-\frac{1}{\sigma},\sigma}} \cong \|u\|_{(X,D(A^{-1}))_{1-\frac{1}{\sigma},\sigma}}$$

$\square$

7.4 Local well-posedness in E_σ

In this section, we prove that the initial value problem (7.1.1) is locally well-posed for $u \in E_\sigma$. We make the same assumptions as in the previous section, that is, A is invertible and X admits a basis of eigenvectors of A.

Remark 7.4.1. The semigroup $\{S_\theta(t)\}_{t\geqslant 0}$ is defined only on X. Given $u \in D(A^{-1})$, we define

$$S_\theta(t)u = S_\theta(t)\left(\sum_{k>0} a_k\phi_k\right) = \sum_{k>0} a_k e^{-e^{i\theta}\lambda_k t}\phi_k, \quad t \geqslant 0.$$

Lemma 7.4.1. *For any $f \in L^1((0,T),E_\sigma)$, define $F[f](t) = \int_0^t S_\theta(t-s)f(s)ds$. Then $F[f] \in L^\infty((0,T),E_\sigma) \cap L^\sigma((0,T),E_\infty)$ and one has the estimates*

$$\|F[f]\|_{L^\infty((0,T),E_\sigma)} \lesssim \|f\|_{L^1((0,T),E_\sigma)} \tag{7.4.1}$$

and

$$\|F[f]\|_{L^\sigma((0,T),E_\infty)} \lesssim \|f\|_{L^1((0,T),E_\sigma)}. \tag{7.4.2}$$

Proof. Since

$$\begin{aligned}\|S_\theta(t-s)f(s)\|_{\dot{E}_\sigma}^\sigma &= \int_0^\infty \|S_\theta(\tau+t-s)f(s)\|^\sigma d\tau = \frac{1}{\cos\theta}\int_{(t-s)\cos\theta}^\infty \|e^{\tau\Delta}f(s)\|^\sigma d\tau \\ &\leqslant \frac{1}{\cos\theta}\|f(s)\|_{\dot{E}_\sigma}^\sigma,\end{aligned}$$

we have

$$\|F[f]\|_{L^\infty((0,T),\dot{E}_\sigma)} \lesssim \sup_{t\in[0,T]}\int_0^t \|S_\theta(t-s)f(s)\|_{\dot{E}_\sigma}ds \lesssim \|f\|_{L^1((0,T),E_\sigma)}.$$

Moreover,

$$\|F[f]\|_{L^\infty((0,T),D(A^{-1}))} \lesssim \sup_{t\in[0,T]}\int_0^t \|S_\theta(t-s)f(s)\|_{D(A^{-1})}ds \lesssim \|f\|_{L^1((0,T),D(A^{-1}))}.$$

This proves the first inequality. For the second, recall that $E_\infty = X$. Then, for any $\phi \in L^{\sigma'}(0,T)$,

$$\begin{aligned}&\int_0^T \left\|\int_0^t S_\theta(t-s)f(s)ds\right\|\phi(t)dt \lesssim \int_0^T\int_0^t \|S_\theta(t-s)f(s)\|ds|\phi(t)|dt \\ &\lesssim \int_0^T\int_s^T \|S_\theta(t-s)f(s)\||\phi(t)|dtds \\ &\lesssim \int_0^T\left(\int_s^T \|S_\theta(t-s)f(s)\|^\sigma dt\right)^{\frac{1}{\sigma}}\left(\int_s^T |\phi(t)|^{\sigma'}dt\right)^{\frac{1}{\sigma'}}ds \\ &\lesssim \int_0^T\left(\int_0^{T-s}\|S_\theta(t)f(s)\|^\sigma dt\right)^{\frac{1}{\sigma}}ds\|\phi\|_{L^{\sigma'}(0,T)} \lesssim \|f\|_{L^1((0,T),E_\sigma)}\|\phi\|_{L^{\sigma'}(0,T)}.\end{aligned}$$

Therefore, by duality,

$$\|F[f]\|_{L^\sigma((0,T),E_\infty)} \lesssim \|f\|_{L^1((0,T),E_\sigma)}.$$

□

Proposition 7.4.2. *For any $u_0 \in E_\sigma$, define $T = T(u_0)$ as in definition 7.2.1 and let $u \in C([0,T), E_\sigma) \cap L^\sigma([0,T), E_\infty)$ be defined by* (7.2.1)*. Then u is the unique (maximal) solution of* (7.1.2) *in this class.*

REMARK 7.4.2. For any $\sigma \geqslant 1$ and $u_0 \in E_\sigma$, the formula (7.2.1) clearly defines a solution of (7.1.2) in the class $C([0,T), E_\sigma) \cap C^1((0,T), E_\infty)$. From the regularization properties of the semigroup, if two solutions coincide at $t_0 > 0$, they are equal for any $t > t_0$. This does not guarantee *a priori* that the uniqueness of solution for $t \geqslant 0$, which is the fact one must prove.

Proof. Let $u_0 \in E_\sigma$ and let u, v be two solutions of (7.1.2) with initial data u_0 in the class $C([0,T), E_\sigma) \cap L^\sigma([0,T), E_\infty)$. If one proves uniqueness for small T, the regularization property of the semigroup will prove uniqueness for any $T > 0$. Therefore, given $\epsilon > 0$, we suppose that T is small enough so that

$$\|u\|_{L^\infty((0,T),E_\sigma)}, \|v\|_{L^\infty((0,T),E_\sigma)} \leqslant 2\|u_0\|_{E_\sigma}, \quad \|u\|_{L^\sigma((0,T),E_\infty)}, \|v\|_{L^\sigma((0,T),E_\infty)} \leqslant \epsilon.$$

Then

$$\begin{aligned}
\|\|u\|^\sigma u - \|v\|^\sigma v\|_{L^1((0,T),E_\sigma)} \lesssim & \int_0^T \|u\|^\sigma \|u - v\|_{E_\sigma} + \|v\|_{E_\sigma} \left|\|u\|^\sigma - \|v\|^\sigma\right| \\
\lesssim & \epsilon \|u - v\|_{L^\infty((0,T),E_\sigma)} + \|u_0\|_{E_\sigma} \int_0^T \left|\|u\|^\sigma - \|v\|^\sigma\right| \\
\lesssim & \epsilon \|u - v\|_{L^\infty((0,T),E_\sigma)} + \|u_0\|_{E_\sigma} \int_0^T \left(\|u\|^{\sigma-1} + \|v\|^{\sigma-1}\right) \|u - v\|
\end{aligned}$$

Using Hölder's inequality and Young's inequality,

$$\begin{aligned}
\|\|u\|^\sigma u - \|v\|^\sigma v\|_{L^1((0,T),E_\sigma)} \lesssim & \epsilon \|u - v\|_{L^\infty((0,T),E_\sigma)} \\
& + \|u_0\|_{E_\sigma} \left(\int_0^T \|u\|^\sigma + \|v\|^\sigma\right) \|u - v\|_{L^\sigma((0,T),E_\infty)} \\
\lesssim & \epsilon \|u - v\|_{L^\infty((0,T),E_\sigma)} + \epsilon \|u_0\|_{E_2} \|u - v\|_{L^\sigma((0,T),E_\infty)}.
\end{aligned}$$

On the other hand, by (7.4.1) and (7.4.2),

$$\begin{aligned}
\|u - v\|_{L^\infty((0,T),E_\sigma)} + \|u - v\|_{L^\sigma((0,T),E_\infty)} & \lesssim \|\|u\|^\sigma u - \|v\|^\sigma v\|_{L^1((0,T),E_\sigma)} \\
& \lesssim \epsilon \left(\|u - v\|_{L^\infty((0,T),E_\sigma)} + \|u - v\|_{L^\sigma((0,T),E_\infty)}\right),
\end{aligned}$$

Therefore, for small ϵ, one obtains $u = v$ in $L^\infty((0,T), E_2) \cap L^2((0,T), E_\infty)$, which concludes the proof. $\square$

7.5 Application to nonlinear heat equations

In this brief section, we show how the abstract evolution problem (7.1.1) gives some information on the solutions of some nonlinear heat equations. We explain the general idea: suppose that $u : [0,T] \times \Omega \to \mathbb{R}$ is a smooth solution of an evolution problem of the form

$$\begin{cases} u_t - \Delta u = N(u)u \\ u(0) = u_0, \quad u\big|_{\partial\Omega} = 0. \end{cases}, \tag{7.5.1}$$

where $N(u)$ can either be local (*i.e.*, $N(u)(x) = N(u(x))$) or a nonlocal operator. In any case, we suppose that, for some $\sigma > 0$,

$$0 \leqslant N(u)(x) \leqslant \|u\|^\sigma, \quad x \in \Omega, \ u \geqslant 0. \tag{7.5.2}$$

The idea is to bound solutions of (7.5.1) using solutions of (7.1.1). In this way, one may obtain directly global existence results for small data. To that end, we must also assume that

$$\|u\| < \|u + w\|, \quad u, w \geqslant 0, \ w \not\equiv 0. \tag{7.5.3}$$

Lemma 7.5.1. *Let $u \geqslant 0$ be a smooth solution of* (7.5.1) *with initial data u_0. Let v be the solution of*

$$v_t - \Delta v = \|v\|^\sigma v, \quad v(0) = u_0.$$

Then $u \leqslant v$ for as long as both exist.

Proof. Take $w_0 \geqslant 0$, $w_0 \not\equiv 0$ and consider, for any $\epsilon > 0$, $w^\epsilon = \epsilon S(t) w_0$. Let v^ϵ be the solution of

$$v_t^\epsilon - \Delta v^\epsilon = \|v^\epsilon\|^\sigma v^\epsilon, \quad v^\epsilon(0) = u_0 + \epsilon w_0.$$

Since $\|u_0\| < \|v_0^\epsilon\|$, there exists a maximal interval $[0, t_0]$ such that

$$\|u(t)\| \leqslant \|v^\epsilon(t)\|, \quad 0 < t \leqslant t_0.$$

One has

$$(u - v^\epsilon + w^\epsilon)_t - \Delta(u - v^\epsilon + w^\epsilon) = N(u)u - \|v^\epsilon\|^\sigma v^\epsilon.$$

Multiplying this equation by $(u - v^\epsilon + w^\epsilon)^+$ and integrating in Ω,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |(u - v^\epsilon + w^\epsilon)^+|^2 &\leqslant \int (N(u) - \|v^\epsilon\|^\sigma) u (u - v^\epsilon + w^\epsilon)^+ \\ &+ \|v^\epsilon\|^\sigma \int (u - v^\epsilon)(u - v^\epsilon + w^\epsilon)^+ \leqslant \|v^\epsilon\|^\sigma \int |(u - v^\epsilon + w^\epsilon)^+|^2, \end{aligned}$$

where we used (7.5.2). Since $(u_0 - v_0^\epsilon + \epsilon w_0)^+ \equiv 0$, we conclude that $u \leqslant v^\epsilon - w^\epsilon$ on $[0, t_0]$. From (7.5.3), $\|u\| < \|u + w^\epsilon\| \leqslant \|v^\epsilon\|$ on $[0, t_0]$, and so t_0 is the maximal time of existence of both u and v^ϵ. The result follows by taking $\epsilon \to 0$. ☐

Corollary 7.5.2. *([76]) Fix Ω an open bounded subset of $\mathbb{R}^N$. Let $u \in C([0,T), L^\infty(\Omega))$ be a maximal weak solution of*

$$u_t - \Delta u = |u|^\sigma u, \quad u(0) = u_0 \geqslant 0, \ u\big|_{\partial\Omega} = 0. \tag{7.5.4}$$

Then, if for some $T_0 > 0$,

$$\int_0^{T_0} \|e^{t\Delta}u_0\|_\infty^\sigma dt \leqslant \frac{1}{\sigma},$$

then $T > T_0$. In particular, if

$$\int_0^{\infty} \|e^{t\Delta}u_0\|_\infty^\sigma dt \leqslant \frac{1}{\sigma},$$

then u is globally defined.

Proof. We take $X = L^\infty(\Omega)$ (observe that Proposition 7.2.2 is still applicable when X is simply a Banach space and A is a m-dissipative operator). Applying Lemma 7.5.1, u is bounded by the solution v of (7.1.1) with initial data u_0. Since v is defined up to time T_0 (cf. Proposition 7.2.2), the blow-up alternative for u implies $T > T_0$. □

Remark 7.5.1. The previous corollary is a sufficient condition for existence of solutions on $[0, T]$. A necessary condition is given in [77]: if the solution of (7.5.4) is defined up to time T, then

$$\sup_{t\in[0,T]} t\|e^{t\Delta}u_0\|_\infty^\sigma \leqslant \frac{1}{\sigma}.$$

It is interesting that the limiting constant is the same in both conditions. The gap between the two conditions comes from the inequality

$$\sup_{t\in[0,T]} t\|e^{t\Delta}u_0\|_\infty^\sigma \leqslant \int_0^T \|e^{t\Delta}u_0\|_\infty^\sigma dt.$$

Corollary 7.5.3. *Fix $\rho \in L^2_w(\mathbb{R}^N)$ positive, that is,*

$$\|\rho\|^2_{L^2_w} := \sup_{t>0} t^2|\{\rho > t\}| < \infty, \quad \rho \geqslant 0.$$

Let $u \in C([0, T), L^2(\mathbb{R}^N))$ be a maximal weak solution of

$$u_t - \Delta u = (\rho * u)^\sigma u, \quad u(0) = u_0 \geqslant 0.$$

Then, if

$$\int_0^{\infty} \|e^{t\Delta}u_0\|_2^\sigma dt \lesssim \frac{1}{\sigma},$$

then $T = \infty$.

Proof. One simply takes $X = L^2(\mathbb{R}^N)$ and applies lemma 7.5.1, noticing that, by Young's inequality,

$$\|\rho * u\|_\infty \lesssim \|\rho\|_{L^2_w}\|u\|_2.$$

□

Part III

New results for the nonlinear Schrödinger equation

Chapter 8

Spatial plane waves and the nonlinear Schrödinger equation

This chapter is devoted to the theory of plane waves and its implications on dynamics of the nonlinear Schrödinger equation. Even though our interest is focused on the (NLS), the ideas and arguments of this section should be applicable to other evolution problems, such as the wave equation, the heat equation, the Zakharov-Kuznetzov equation, and so on.

Let us briefly recall what has been done in Section 5.4: in the context of the hyperbolic nonlinear Schrödinger equation

$$iu_t + u_{xx} - \Delta_{\mathbf{y}} u + \lambda |u|^\sigma u = 0, \quad u : [0,T) \times \mathbb{R}^d \to \mathbb{C},$$

one may look for spatial plane waves in $\mathbb{R}^d$, $d \geqslant 2$, that is,

$$u(t,x,y,\mathbf{z}) = f(t, x-cy, \mathbf{z}).$$

We say that f is the *profile* and c is the *speed* of the wave. These solutions satisfy a nonlinear Schrödinger equation in dimension one, and so their behaviour can be well-understood. Evidently, these solutions are not integrable in $\mathbb{R}^2$. However, assuming that $f(t,\cdot) \in H^s(\mathbb{R}^{d-1})$, $2s > d-1$, it is clear that they are L^∞ solutions.

We then considered the space of spatial plane waves in $\mathbb{R}^2$

$$X_c = \left\{ u \in L^1_{loc}(\mathbb{R}^2) : \exists f \in H^2(\mathbb{R}) : u(x,y) = f(x-cy) \ a.e. \right\}$$

and looked for a local well-posedness result over

$$E = H^1(\mathbb{R}^2) \oplus X_c.$$

Due to the lack of decay of elements in X_c, the sum is indeed a direct sum. Then the initial value problem for $u_0 = v_0 + \phi_0 \in E$ was proven to be equivalent to the initial value problem for the system

$$\begin{cases} iv_t + v_{xx} - v_{yy} + \lambda|v+\phi|^\sigma(v+\phi) - \lambda|\phi|^\sigma\phi = 0, \ \phi(t,x,y) = f(t,x-cy) \\ v(0) = v_0 \\ if_t + (1-c^2)f_{zz} + \lambda|f|^\sigma f = 0 \\ f(0) = f_0 \end{cases}$$

where f_0 is the profile of ϕ_0. Since the second equation is independent of v, one could solve it and introduce the solution onto the first equation. Thus, to prove local well-posedness on E, it sufficed to show an H^1 local well-posedness result for the first equation. This was achieved by making the crucial observation that the nonlinear terms lie in H^{-1} (even though they involve ϕ, which is only in L^∞).

With such a local well-posedness result, we considered the following situation: take a spatial plane wave ϕ with initial data ϕ_0, and assume that ϕ is globally defined. If one introduces an H^1 perturbation on the initial data, $\phi_0 + \epsilon v_0$, the corresponding solution is given by $u = \phi + v_\epsilon$ (since ϕ only depends on the plane wave part of the initial data). Then a natural question is the following:

Stability problem: if ϵ is sufficiently small, does u stay close to ϕ?

This question corresponds to a global existence result for v that ensures that v stays small for all times. Results of such nature are indeed valid for (NLS) in H^1 for large σ (see [15], [39], [40], [43], [59]). Looking at the equation for v, two types of problems may arise: the presence of lower order terms in v and the lack of integrability of ϕ. Despite these difficulties, we proved the result for any $\sigma \geqslant 4$ even.

In light of these results, we make two observations:

- This framework can be applied to other equations, such as the (NLS): the proofs rely solely on generic properties of the semigroup;
- This process can be iterated: instead of having a single plane wave with speed c, one could have a collection of plane waves, each one with a distinct speed.

This chapter is devoted to the development of these ideas in the context of the nonlinear Schrödinger equation in $\mathbb{R}^2$

$$iu_t + \Delta u + \lambda |u|^\sigma u = 0. \tag{NLS}$$

Before we enter into the technicalities of the proofs, we give a brief summary of the contents of this chapter.

Take a sequence of wave speeds $\underline{c} = \{c_n\}_{n\in\mathbb{N}}$, with $c_i \neq c_j$, $i \neq j$, and consider the space

$$X_{\underline{c}} = \Big\{\phi \in L^1_{loc}(\mathbb{R}^2) : \phi(x,y) = \sum_{n\geqslant 1} f_n(x - c_n y), ((1+c_n^2)f_n)_{n\in\mathbb{N}} \in l^1(H^2(\mathbb{R}))\Big\}.$$

endowed with the induced norm (which, as we will check, is well-defined). As before, we look for a local well-posedness result for the (NLS) in

$$E_{\underline{c}} = H^1(\mathbb{R}^2) \oplus X_{\underline{c}},$$

which turns out to be equivalent to a local well-posedness result for the infinite system

$$\begin{cases} iv_t + \Delta v + \lambda|v+\phi|^\sigma(v+\phi) - \sum_{n\geqslant 1} \lambda|\phi_n|^\sigma\phi_n = 0, \quad \phi_n(t,x,y) = f_n(t, x - c_n y) \\ i((f_n)_t + (1+c_n^2)(f_n)_{zz} + \lambda|f_n|^\sigma f_n = 0, \ n \in \mathbb{N} \end{cases}$$

The equations for the profiles f_n are solved using the $H^2(\mathbb{R})$ local well-posedness result (here, one must be careful to ensure that *all* profiles exist up to some time $T > 0$). Once again, it remains to prove that one may solve the equation for v in $H^1(\mathbb{R}^2)$, which amounts to check that the nonlinear terms are in $H^{-1}(\mathbb{R}^2)$. Under the additional hypothesis $\sigma \geqslant 1$, this can be shown to be true: heuristically, if one develops the nonlinear terms, one obtains some terms with v, which are well-behaved, and some products of diferent ϕ_n's. Remarkably, these terms lie in $L^2(\mathbb{R}^2)$: take, for example, $\phi_j\phi_k$, $j \neq k$. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^2} |\phi_j(x,y)|^2|\phi_k(x,y)|^2 dxdy\right)^{1/2} &= \left(\int_{\mathbb{R}^2} |f_j(x-c_jy)|^2|f_k(x-c_ky)|^2 dxdy\right)^{1/2}\\ &= \frac{1}{|c_j-c_k|^{1/2}}\left(\int_{\mathbb{R}^2} |f_j(w)|^2|f_k(z)|^2 dwdz\right)^{1/2}\\ &\leqslant \frac{1}{|c_j-c_k|^{1/2}}\|f_j\|_{L^2}\|f_k\|_{L^2}. \end{aligned} \tag{8.0.1}$$

However, since one has an infinite sum of such terms, one must be able to control

$$\sum_{j\neq k} \frac{1}{|c_j-c_k|^{1/2}}\|f_j\|_{L^2}\|f_k\|_{L^2}.$$

The above quantity turns out to be preserved by the flow of the infinite system, since one has conservation of the L^2 norm of each f_n, and therefore it can controlled by the initial data. For similar but more technical reasons, we shall also need to control products of the form $\phi_j\nabla\phi_k$, $j \neq k$:

$$\left(\int_{\mathbb{R}^2} |\phi_j|^2|\nabla\phi_k|^2\right)^{1/2} \leqslant \frac{(1+c_k^2)^{1/2}}{|c_j-c_k|^{1/2}}\|f_j\|_{L^2}\|\partial_z f_k\|_{L^2}.$$

Hence we shall restrict ourselves to $A_{\underline{c}} = H^1(\mathbb{R}^2)\oplus Y_{\underline{c}}$, where

$$Y_{\underline{c}} = \left\{\phi \in X_{\underline{c}} : \sum_{j\neq k}\frac{\|f_j\|_{L^2}\|f_k\|_{L^2}}{|c_j-c_k|^{1/2}} < \infty,\ \sum_{j\neq k}\frac{(1+c_k^2)^{1/2}}{|c_j-c_k|^{1/2}}\|f_j\|_{L^2}\|\partial_z f_k\|_{L^2} < \infty\right\}.$$

One finally concludes that it is possible to solve the equation for v and arrives to the local well-posedness result over $A_{\underline{c}}$. The stability property is again true for any $\sigma \geqslant 4$ even, under the additional hypothesis that the initial profiles of the plane waves are small enough (recall, however, that the plane wave still has infinite mass and energy). The smallness condition appears naturally because of the interference between waves with different speeds. These results are stated precisely and shown in Section 8.1.

In the remainder of the chapter, we replace the infinite sum of plane waves with an integration over a *continuum of wave speeds*. That is, we consider

$$X = \left\{\phi \in L^1_{loc}(\mathbb{R}^2) : \phi(x,y) = \int_{\mathbb{R}} f(x-cy,c)dc, f \in L^1_c(\mathbb{R}; H^1_z(\mathbb{R})) \cap L^\infty_c(\mathbb{R}, L^2_z(\mathbb{R}))\right\},$$

endowed with the norm $\|\phi\| = \|f\|_{L^1_c(H^1_z)} + \|f\|_{L^\infty_c(L^2_z)}$. One can actually define an integral transform, the *plane wave transform*, mapping functions in two *speed* variables to functions in two *physical* variables

$$(Tf)(x,y) = \int f(x-cy, c)dc.$$

As we shall see, the properties of the continuous case differ from those of the numerable case. On a first step, the focus must be directed to understand how the plane wave transform works (see Section 8.2). Among other properties, we highlight the following:

- $Tf \in L^p(\mathbb{R}^2)$, $p \geqslant 2$ under suitable conditions on f;
- $Tf \notin L^2(\mathbb{R}^2)$ if $f \in C_0(\mathbb{R}^2)$ is positive;
- The convolution of two functions, the Fourier transform and the Laplace transform may be obtained using the plane wave transform transform;
- Several classical linear equations, such as the heat equation, the Schrödinger equation and the wave equation, may be solved by means of this transform.

One should regard the theory of plane waves in analogy to the Fourier series and transform: one starts with a simple periodic function with a given frequency, superposes a numerable family of functions with different frequencies to arrive to the Fourier series and passes to the continuous case to obtain the Fourier transform, where one also has the concepts of physical and frequency variables.

Evidently, our construction is not the same as the Fourier one and many properties will differ. One aspect is that, while the Fourier construction makes all sense in one variable (and its multidimensional version is simply the application of the one-dimensional case to each variable), the plane wave construction needs (at least) a two-dimensional setting. In another perspective, the Fourier transform is based on the solutions of the ODE

$$u''(x) + k^2 u(x) = 0,$$

while the plane wave theory is based on solutions of the transport equation

$$u_y(x,y) + cu_x(x,y) = 0.$$

With this new transform in hand, we try to obtain some results in the spirit of the numerable case. That is, look for local well-posedness and stability results in $E = H^1(\mathbb{R}^2) + X$ (this sum is *not* a direct sum). This is done in Section 8.4. The main difference is that now the (NLS) may be decoupled into

$$\begin{cases} iv_t + v_{xx} + v_{yy} + \lambda|v+\phi|^\sigma(v+\phi) = 0 \\ v(0) = v_0 \\ i\phi_t + \phi_{xx} + \phi_{yy} = 0 \\ \phi(0) = \phi_0 \end{cases}.$$

Surprisingly, when one passes to the continuous case, there exists a decoupling such that the equation for the plane wave component is linear. The existence of solution for this system is trivial. However, since E is not a direct sum, one cannot prove the equivalence of (NLS) and this system directly: it is not clear that different decompositions of an initial data $u_0 = v_0 + \phi_0 = v_1 + \phi_1 \in E$ do not give rise to different solutions. This problematic can be solved by proving directly a uniqueness result over E and so one is able to prove a local well-posedness result over E.

Regarding the stability property, the proof of the numerable case is easily extendible to the continuous case: for $\sigma \geqslant 4$ even and if the profile f is small in some suitable norms, then the stability of the plane wave component was valid. Notice that, since the plane wave component can lie also in $H^1(\mathbb{R}^2)$, one may construct H^1 global solutions with arbitrarily large norms. This observation shows that the theory of spatial plane waves, apart from its intrinsic interest, can also provide some new results for the classical H^1 theory for the (NLS).

8.1 The numerable case

Let us recall the numerable construction. Fix a sequence of wave speeds $\underline{c} = \{c_n\}_{n\in\mathbb{N}}$, with $c_i \neq c_j$, $i \neq j$. Define the spaces

$$X_{\underline{c}} = \left\{\phi \in L^1_{loc}(\mathbb{R}^2) : \phi(x,y) = \sum_{n\geqslant 1} f_n(x - c_n y), ((1 + c_n^2)f_n)_{n\in\mathbb{N}} \in l^1(H^2(\mathbb{R})), \right\}$$

and

$$Z_{\underline{c}} = \left\{\phi \in L^1_{loc}(\mathbb{R}^2) : \phi(x,y) = \sum_{n\geqslant 1} f_n(x - c_n y), (f_n)_{n\in\mathbb{N}} \in l^1(L^2(\mathbb{R}))\right\}.$$

Lemma 8.1.1. *If $\underline{f} \in l^1(L^2(\mathbb{R}))$ is such that*

$$\phi(x,y) = \sum_{n\geqslant 1} f_n(x - c_n y) = 0, \quad x, y \in \mathbb{R}^2,$$

then $\underline{f} \equiv 0$.

Proof. Fix $k \in \mathbb{N}$, $h \in \mathbb{R}$ and define

$$\phi_h(x) = \phi(x + c_k h, h) = f_k(x) + \sum_{n\geqslant 1, n\neq k} f_n(x + (c_k - c_n)h).$$

Arguing by contradiction, suppose that there exists an open interval $]a, b[$ such that

$$\|f_k\|_{L^2(]a,b[)} = \delta > 0.$$

Since $\underline{f} \in l^1(L^2(\mathbb{R}))$, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n\geqslant n_0} \|f_n\|_{L^2} < \delta/2.$$

On the other hand, for h large enough, one has

$$\sum_{n=1,n\neq k}^{n_0} \|f_n(\cdot + (c_k - c_n)h)\|_{L^2(]a,b[)} < \delta/2.$$

Hence

$$\delta = \|f_k\|_{L^2(]a,b[)} \leqslant \sum_{n\geqslant 1,n\neq k} \|f_n(\cdot + (c_k - c_n)h)\|_{L^2(]a,b[)} < \delta,$$

which is impossible. □

Since each element of $X_{\underline{c}}$ and $Z_{\underline{c}}$ may be represented in a unique way, we define the norms of these spaces as the norm induced by the profile space:

$$\|\phi\|_{X_{\underline{c}}} := \sum_{n\geqslant 1} \|(1 + c_n^2)f_n\|_{H^2}, \quad \|\phi\|_{Z_{\underline{c}}} := \sum_{n\geqslant 1} \|f_n\|_{L^2}.$$

Lemma 8.1.2. *Consider $H^{-1}(\mathbb{R}^2)$ and $Z_{\underline{c}}$ as subspaces of $\mathcal{D}'(\mathbb{R}^2)$. Then $H^{-1}(\mathbb{R}^2) \cap Z_{\underline{c}} = \{0\}$.*

Proof. Suppose that $\underline{f} \in l^1(L^2(\mathbb{R}))$ is such that the function ϕ defined by

$$\phi(x, y) = \sum_{n\geqslant 1} f_n(x - c_n y)$$

is in $H^{-1}(\mathbb{R}^2)$. Fix $k \in \mathbb{N}$ and define

$$\phi_h(x, y) = \phi(x + c_k h, y + h) = f_k(x - c_k y) + \sum_{n\geqslant 1,n\neq k} f_n(x - c_n y + (c_k - c_n)h).$$

Given $\psi \in C_0^\infty(\mathbb{R}^2)$, it is easy to check (see Lemma 5.4.1) that, since $\phi \in H^{-1}(\mathbb{R}^2)$,

$$\langle \phi_h, \psi\rangle_{H^{-1}\times H^1} \to 0, \quad h \to \infty.$$

Through a similar argument to that of the previous proof, one may check that, when $h \to \infty$,

$$\int \left(f_k(x - c_k y) + \sum_{n\geqslant 1,n\neq k} f_n(x - c_n y + (c_k - c_n)h) \right) \psi(x, y)dxdy$$
$$\to \int f_k(x - c_k y)\psi(x, y)dxdy.$$

Hence one has necessarily

$$\int f_k(x - c_k y)\psi(x, y)dxdy = 0, \forall \psi \in C_0^\infty(\mathbb{R}^2)$$

and so $f_k \equiv 0$. Since $k \in \mathbb{N}$ is arbitrary, one concludes that $\phi \equiv 0$. □

Define the following subspaces of $\mathcal{D}'(\mathbb{R}^2)$:

$$E_{\underline{c}} = H^1(\mathbb{R}^2) \oplus X_{\underline{c}}, \quad E'_{\underline{c}} = H^{-1}(\mathbb{R}^2) \oplus Z_{\underline{c}}. \tag{8.1.1}$$

Notice that it follows from the previous lemma that these sums are indeed direct sums. Moreover, consider $A_{\underline{c}} = H^1(\mathbb{R}^2) \oplus Y_{\underline{c}}$, where

$$Y_{\underline{c}} = \left\{ \phi \in X_{\underline{c}} : \sum_{j\neq k} \frac{\|f_j\|_{L^2}\|f_k\|_{L^2}}{|c_j - c_k|^{1/2}} < \infty,\ \sum_{j\neq k} \frac{(1+c_k^2)^{1/2}}{|c_j - c_k|^{1/2}} \|f_j\|_{L^2}\|\partial_z f_k\|_{L^2} < \infty \right\}.$$

Lemma 8.1.3. *Fix $\sigma \geqslant 1$. Take $v \in H^1(\mathbb{R}^2)$ and $\phi \in Y_{\underline{c}}$. Writing*

$$\phi(x,y) = \sum_{n\geqslant 1} f_n(x - c_n y),$$

define $\phi_n(x,y) = f_n(x - c_n y)$ and

$$g = \sum_{n\geqslant 1} |\phi_n|^\sigma \phi_n$$

Then $g \in Z_{\underline{c}}$, $|v+\phi|^\sigma(v+\phi) - g \in L^{\frac{\sigma+2}{\sigma+1}} + L^2$ and

$$\left\||v+\phi|^\sigma(v+\phi) - g\right\|_{L^{\frac{\sigma+2}{\sigma+1}}+L^2} \lesssim \|v\|_{H^1}^{\sigma+1} + \|v\|_{H^1}\|\phi\|_{X_{\underline{c}}}^\sigma + \|\phi\|_{X_{\underline{c}}}^{\sigma-1} \sum_{j\neq k} \frac{\|f_j\|_{L^2}\|f_k\|_{L^2}}{|c_j - c_k|^{1/2}}.$$

Furthermore, if $\nabla v \in L^{\sigma+2}$, $\nabla(|v+\phi|^\sigma(v+\phi) - g) \in L^{\frac{\sigma+2}{\sigma+1}} + L^2$ and

$$\begin{aligned}\left\|\nabla(|v+\phi|^\sigma(v+\phi) - g)\right\|_{L^{\frac{\sigma+2}{\sigma+1}}+L^2} &\lesssim \|v\|_{L^{\sigma+2}}^\sigma \|\nabla v\|_{L^{\sigma+2}} + \|v\|_{H^1}^\sigma \|\phi\|_{X_{\underline{c}}} + \|v\|_{H^1}\|\phi\|_{X_{\underline{c}}}^\sigma \\ &\quad + \|\phi\|_{X_{\underline{c}}}^{\sigma-1} \sum_{j\neq k} \frac{(1+c_k^2)^{1/2}}{|c_j - c_k|^{1/2}} \|f_j\|_{L^2}\|\nabla f_k\|_{L^2}.\end{aligned}$$

Proof. Since, for any $k \in \mathbb{N}$,

$$\|\phi_k\|_{L^\infty} \leqslant \sum_{n\geqslant 1} \|\phi_n\|_{L^\infty} \leqslant \sum_{n\geqslant 1} \|f_n\|_{L^\infty} \leqslant \|\phi\|_{X_{\underline{c}}}, \tag{8.1.2}$$

one has

$$\|g\|_{Z_{\underline{c}}} \leqslant \left(\sup_k \|\phi_k\|_\infty^\sigma\right) \left\| \sum_{n\geqslant 1} |\phi_n| \right\|_{Z_{\underline{c}}} \leqslant \|\phi\|_{X_{\underline{c}}}^\sigma \left\| \sum_{n\geqslant 1} |\phi_n| \right\|_{Z_{\underline{c}}} \lesssim \|\phi\|_{X_{\underline{c}}}^{\sigma+1}$$

and so $g \in Z_{\underline{c}}$. For the second part of the result, recall the classical estimates

$$||a|^\sigma a - |b|^\sigma b| \leqslant (\sigma+1)(|a|^\sigma + |b|^\sigma)|a-b|,\ \ a, b \in \mathbb{C}$$

and

$$||a+b|^\sigma(a+b)-|a|^\sigma a-|b|^\sigma b|\lesssim |a|^\sigma|b|+|b|^\sigma|a|,\ a,b\in\mathbb{C}.$$

Applying the triangular inequality $N+1$ times,

$$\begin{aligned}\left|\left|v+\sum_{k\geqslant 1}\phi_k\right|^\sigma\left(v+\sum_{k\geqslant 1}\phi_k\right)-g\right| &\leqslant \left|\left|v+\sum_{k\geqslant 1}\phi_k\right|^\sigma\left(v+\sum_{k\geqslant 1}\phi_k\right)-\left|\sum_{k\geqslant 1}\phi_k\right|^\sigma\left(\sum_{k\geqslant 1}\phi_k\right)\right|\\ &+\sum_{j=1}^N\left|\left|\sum_{k\geqslant j}\phi_k\right|^\sigma\left(\sum_{k\geqslant j}\phi_k\right)-\left|\sum_{k\geqslant j+1}\phi_k\right|^\sigma\left(\sum_{k\geqslant j+1}\phi_k\right)-|\phi_j|^\sigma\phi_j\right|\\ &+\left|\left|\sum_{k\geqslant N+1}\phi_k\right|^\sigma\left(\sum_{k\geqslant N+1}\phi_k\right)-\sum_{k\geqslant N+1}|\phi_k|^\sigma\phi_k\right|\end{aligned}$$

From (8.1.2), the limit of the last term as $N\to\infty$ is 0. Hence

$$\begin{aligned}\left|\left|v+\sum_{k\geqslant 1}\phi_k\right|^\sigma\left(v+\sum_{k\geqslant 1}\phi_k\right)-g\right| &\leqslant \left|\left|v+\sum_{k\geqslant 1}\phi_k\right|^\sigma\left(v+\sum_{k\geqslant 1}\phi_k\right)-\left|\sum_{k\geqslant 1}\phi_k\right|^\sigma\left(\sum_{k\geqslant 1}\phi_k\right)\right|\\ &+\sum_{j\geqslant 1}\left|\left|\sum_{k\geqslant j}\phi_k\right|^\sigma\left(\sum_{k\geqslant j}\phi_k\right)-\left|\sum_{k\geqslant j+1}\phi_k\right|^\sigma\left(\sum_{k\geqslant j+1}\phi_k\right)-|\phi_j|^\sigma\phi_j\right|\\ &\lesssim\left(|v|^\sigma+\left|\sum_{k\geqslant 1}\phi_k\right|^\sigma\right)|v|+\sum_{j\geqslant 1}\left(\left|\sum_{k\geqslant j+1}\phi_k\right|^\sigma|\phi_j|+\left|\sum_{k\geqslant j+1}\phi_k\right||\phi_j|^\sigma\right)\\ &\lesssim|v|^{\sigma+1}+\sum_{k\geqslant 1}\|\phi_k\|_{L^\infty}^\sigma|v|+\sum_{j\geqslant 1}\sum_{k\geqslant j+1}\left(\|\phi_k\|_{L^\infty}^{\sigma-1}+\|\phi_j\|_{L^\infty}^{\sigma-1}\right)|\phi_j\phi_k|\end{aligned}$$

The first term, $|v|^{\sigma+1}$, is in $L^{\frac{\sigma+2}{\sigma+1}}$, by Sobolev's injection. The second term is clearly in L^2. Finally, we prove that the third term is also in L^2: recalling (8.0.1),

$$\begin{aligned}\left\|\sum_{j\geqslant 1}\sum_{k\geqslant j+1}\left(\|\phi_k\|_{L^\infty}^{\sigma-1}+\|\phi_j\|_{L^\infty}^{\sigma-1}\right)|\phi_j\phi_k|\right\|_{L^2} &\lesssim\|\phi\|_{X_{\underline{c}}}^{\sigma-1}\sum_{j\geqslant 1}\sum_{k\geqslant j+1}\|\phi_j\phi_k\|_{L^2}\\ &\leqslant\|\phi\|_{X_{\underline{c}}}^{\sigma-1}\sum_{j\neq k}\frac{\|f_j\|_{L^2}\|f_k\|_{L^2}}{|c_j-c_k|^{1/2}}<\infty.\end{aligned}$$

Hence $|v+\phi|^\sigma(v+\phi)-g\in L^{\frac{\sigma+2}{\sigma+1}}+L^2$ and

$$\||v+\phi|^\sigma(v+\phi)-g\|_{L^{\frac{\sigma+2}{\sigma+1}}+L^2}\lesssim\|v\|_{H^1}^{\sigma+1}+\|v\|_{H^1}\sum_{k\geqslant 1}\|\phi_k\|_{L^\infty}^\sigma+\|\phi\|_{X_{\underline{c}}}^{\sigma-1}\sum_{j\neq k}\frac{\|f_j\|_{L^2}\|f_k\|_{L^2}}{|c_j-c_k|^{1/2}}.$$

We now prove the last assertion.

$$\nabla(|w|^\sigma w)=\left(\frac{\sigma}{2}+1\right)|w|^\sigma\nabla w+\frac{\sigma}{2}|w|^{\sigma-2}w^2\nabla\bar{w}.$$

Hence

$$\nabla(|v+\phi|^\sigma(v+\phi)-g) = \left(\frac{\sigma}{2}+1\right)|v+\phi|^\sigma\nabla(v+\phi) + \frac{\sigma}{2}|v+\phi|^{\sigma-2}(v+\phi)^2\nabla\overline{(v+\phi)}$$
$$- \sum_{k\geqslant 1}\left(\frac{\sigma}{2}+1\right)|\phi_k|^\sigma\nabla\phi_k + \frac{\sigma}{2}|\phi_k|^{\sigma-2}\phi_k^2\nabla\overline{\phi_k}$$

and so

$$\begin{aligned}|\nabla(|v+\phi|^\sigma(v+\phi)-g)| \leqslant{}& \left|\left(\frac{\sigma}{2}+1\right)|v+\phi|^\sigma\nabla(v+\phi) - \left(\frac{\sigma}{2}+1\right)|\phi|^\sigma\nabla\phi\right| \\ &+ \left|\frac{\sigma}{2}|v+\phi|^{\sigma-2}(v+\phi)^2\nabla\overline{(v+\phi)} - \frac{\sigma}{2}|\phi|^{\sigma-2}\phi^2\nabla\overline{\phi}\right| \\ &+ \left|\left(\frac{\sigma}{2}+1\right)|\phi|^\sigma\nabla\phi - \sum_{k\geqslant 1}\left(\frac{\sigma}{2}+1\right)|\phi_k|^\sigma\nabla\phi_k\right| \\ &+ \left|\frac{\sigma}{2}|\phi|^{\sigma-2}\phi^2\nabla\overline{\phi} - \sum_{k\geqslant 1}\frac{\sigma}{2}|\phi_k|^{\sigma-2}\phi_k^2\nabla\overline{\phi_k}\right| = I_1+I_2+I_3+I_4.\end{aligned}$$

We now estimate

$$I_1+I_2 \lesssim (|v|^\sigma+|\phi|^\sigma)|\nabla v| + (|v|^{\sigma-1}+|\phi|^{\sigma-1})|v||\nabla\phi|$$

and

$$\begin{aligned}I_3 &\lesssim \left|\sum_{k\geqslant 1}\left(|\phi|^\sigma - |\phi_k|^\sigma\right)\nabla\phi_k\right| \lesssim \sum_{k\geqslant 1}\left(|\phi|^{\sigma-1}+|\phi_k|^{\sigma-1}\right)|\phi-\phi_k||\nabla\phi_k| \\ &\lesssim \|\phi\|_\infty\sum_{j\neq k}|\phi_j||\nabla\phi_k|.\end{aligned}$$

The analogous estimate may be derived for I_4. Since, for any $j\neq k$,

$$\|\phi_j\nabla\phi_k\|^2_{L^2(\mathbb{R}^2)} \leqslant \frac{1+c_k^2}{|c_j-c_k|}\|f_j\|_2^2\|\partial_z f_k\|_2^2,$$

the claimed estimate follows easily. □

Theorem 8.1.4. *Fix $\sigma\geqslant 1$ and $u_0\in A_{\underline{c}}$. Then there exists a unique maximal solution $u\in C([0,T),A_{\underline{c}})\cap C^1((0,T),E'_{\underline{c}})$ (cf.* (8.1.1)*) of (NLS) such that $u(0)=u_0$. Furthermore, if $T<\infty$, then*

$$\|u(t)\|_{E_{\underline{c}}}\to\infty, \quad t\to T.$$

Proof. Step 1. Write $u_0 = v_0+\phi_0$, $v_0\in H^1(\mathbb{R}^2)$, $\phi_0\in Y_{\underline{c}}$,

$$\phi_0(x,y) = \sum_{n\geqslant 1}(f_0)_n(x-c_ny).$$

We claim that, over $A_{\underline{c}}$, the initial value problem for (NLS) is equivalent to the initial value problem

$$\begin{cases} iv_t + \Delta v + \lambda |v+\phi|^\sigma (v+\phi) - \sum_{n\geqslant 1} \lambda |\phi_n|^\sigma \phi_n = 0, & (8.1.3) \\ v(0) = v_0 \\ i(f_n)_t + (1+c_n^2)(f_n)_{zz} + \lambda |f_n|^\sigma f_n = 0, \; n \in \mathbb{N} & (8.1.4) \\ f_n(0) = (f_0)_n \end{cases}$$

where $\phi_n(t,x,y) = f_n(t, x - c_n y)$. Indeed, if v and $(f_n)_{n\in\mathbb{N}}$ are solutions of this problem, it is trivial to check that

$$u(t,x,y) = v(t,x,y) + \sum_{n\geqslant 1} f_n(t, x - c_n y)$$

is a solution of (NLS). On the other hand, suppose that u is a solution of (NLS) with initial condition u_0. Decompose u as $v + \phi$ and write

$$\phi(t,x,y) = \sum_{n\geqslant 1} f_n(t, x - c_n y), \quad g(t,x,y) = \sum_{n\geqslant 1} \left(|f_n|^\sigma f_n\right)(t, x - c_n y)$$

Then

$$iv_t + v_{xx} + v_{yy} + \lambda |v+\phi|^\sigma (v+\phi) - \lambda g = -\left(i\phi_t + \phi_{xx} + \phi_{yy} + \lambda g\right)$$

Since the left hand side is in $H^{-1}(\mathbb{R}^2)$ and the right hand side is in $Z_{\underline{c}}$, by Lemma 8.1.2, both sides must be equal to zero:

$$\begin{cases} iv_t + v_{xx} + v_{yy} + \lambda |v+\phi|^\sigma (v+\phi) - \lambda g = 0 \\ v(0) = v_0 \\ i\phi_t + \phi_{xx} + \phi_{yy} + \lambda g = 0 \\ \phi(0) = \phi_0 \end{cases}.$$

Furthermore, by Lemma 8.1.1, the second equation is equivalent to the infinite system (8.1.4), which proves the claim.

Step 2. We solve the infinite system (8.1.4). For each $n \in \mathbb{N}$, define

$$(h_0)_n(z) = (f_0)_n(\sqrt{1+c_n^2}\, z).$$

Then

$$\|h_n(0)\|_{H^2} \lesssim (1+c_n^2)^{3/4} \|(f_0)_n\|_{H^2}.$$

Since $((1+c_n^2)(f_0)_n) \in l^1(H^2)$, $(1+c_n^2)^{3/4}\|(f_0)_n\|_{H^2} \to 0$ as $n \to \infty$.

Now consider the initial value problem

$$i(h_n)_t + (h_n)_{zz} + \lambda |h_n|^\sigma h_n = 0, h_n(0) = (h_n)_0.$$

It follows from the H^2 local well-posedness results for (NLS) that there exists a time T_n and a unique maximal solution $h_n \in C([0,T_n), H^2(\mathbb{R})) \cap C^1([0,T_n), L^2(\mathbb{R}))$ of the above problem. Moreover, since $\|h_n(0)\|_{H^2} \to 0$, for $n \geqslant n_0$ sufficiently large, $T_n = \infty$. Define the common time of existence,

$$T_\infty = \inf_{n < n_0} T_n.$$

Notice that, since the infimum is taken over a finite set, $T_\infty > 0$. Then each h_n is define on $[0, T_\infty)$. If $T_\infty < \infty$, then, as $t \to T_\infty$,

$$\|h_n(t)\|_{H^2} \to \infty, \text{ for some } n \in \mathbb{N}.$$

Setting

$$f_n(t, \cdot) = h_n\left(t, \frac{\cdot}{\sqrt{1 + c_n^2}}\right), \ 0 \leqslant t < T_\infty,$$

it is clear that the sequence $(f_n)_{n\in\mathbb{N}}$ is the unique solution of (8.1.4) on $[0, T_\infty)$. Moreover, considering that

$$\sup_n \|h_n(t)\|_{H^2} \leqslant \sup_n (1 + c_n^2)^{3/4} \|f_n(t)\|_{H^2} \leqslant \left\| \left((1 + c_n^2) f_n(t)\right)_{n\in\mathbb{N}} \right\|_{l^1(H^2)},$$

if $T_\infty < \infty$, one has

$$\left\| \left((1 + c_n^2) f_n(t)\right)_{n\in\mathbb{N}} \right\|_{l^1(H^2)} \to \infty, \quad t \to T_\infty.$$

Finally, it follows from the conservation of the L^2 norm that

$$\left(\sum_{j\neq k} \frac{\|f_j(t)\|_{L^2} \|f_k(t)\|_{L^2}}{|c_j - c_k|^{1/2}}\right)^{1/2} = \left(\sum_{j\neq k} \frac{\|(f_0)_j\|_{L^2} \|(f_0)_k\|_{L^2}}{|c_j - c_k|^{1/2}}\right)^{1/2}, \ 0 \leqslant t < T_\infty.$$

Moreover, since $\|h_n(0)\|_{H^2} \to 0$, it is not hard to check that

$$\sum_{j\neq k} \frac{(1 + c_k^2)^{1/2}}{|c_j - c_k|^{1/2}} \|f_j(t)\|_{L^2} \|\partial_z f_k(t)\|_{L^2} \leqslant C(T), \quad 0 \leqslant t \leqslant T < T_\infty$$

Setting

$$\phi(t, x, y) = \sum_{n\geqslant 1} f_n(t, x - c_n y),$$

this implies that $\phi \in C([0, T_\infty), Y_{\underline{c}}) \cap C^1((0, T_\infty), Z_{\underline{c}})$.

Step 3. We now solve (8.1.3). The idea here is to use Kato's method (see, for example, Section 4.4 of [12]). Notice that the main problem is that the nonlinearity depends on t and x through ϕ and g. However, this does not pose any difficulty, since ϕ and g work essentially as $W^{1,\infty}$ coefficients. We write $N(v) = |v + \phi|^\sigma (v + \phi) - g$. Any constants involving norms of ϕ will be ommited.

Set $r = \sigma + 2$ and q such that (q, r) is an admissible pair. Given $T, M > 0$, consider the space

$$\mathcal{E} = \Big\{ v \in L^\infty((0, T); H^1(\mathbb{R}^2)) \cap L^q((0, T); W^{1,r}(\mathbb{R}^2)) :$$

$$\|v\|_{L^\infty((0,T);H^1(\mathbb{R}^2))} < M,\ \|v\|_{L^q((0,T);W^{1,r}(\mathbb{R}^2))} < M\Big\}.$$

endowed with the distance

$$d(v,w) = \|v-w\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} + \|v-w\|_{L^q((0,T);L^r(\mathbb{R}^2))}.$$

It is easy to check that $\mathcal{E}$ is a complete metric space. Set

$$\Phi(v) = S_2(t)v_0 + \int_0^t S_2(t-s)N(v)(s)ds.$$

It follows from Strichartz estimates and Lemma 8.1.3 that

$$\|\Phi(v)\|_{L^\infty((0,T),H^1(\mathbb{R}^2))} + \|\Phi(v)\|_{L^q((0,T),W^{1,r}(\mathbb{R}^2))} \lesssim \|v_0\|_{H^1} + (T + T^{\frac{q-q'}{qq'}})(1+M^{\sigma+1})$$

From the estimate

$$|N(v) - N(w)| \lesssim (1 + |\phi|^\sigma + |v|^\sigma + |w|^\sigma)|v-w|$$

one may deduce the usual Lipschitz estimate for Φ:

$$d(\Phi(v), \Phi(w)) \lesssim (T + T^{\frac{q-q'}{qq'}})(1+M^\sigma)d(v,w).$$

Choosing $T, M > 0$ in such a way that $\Phi : \mathcal{E} \to \mathcal{E}$ is a strict contraction mapping, we obtain, by Banach's fixed point theorem, the existence and uniqueness of a local solution v to (8.1.3) with initial data v_0 on $[0, T_{max})$, where $T_{max} = \min\{T_\infty, T_v\}$. The uniqueness property allows the construction of a unique maximal solution defined on $[0, T_v)$, $T_v \leqslant T_\infty$. Furthermore, if $T_v < T_\infty$,

$$\|v(t)\|_{H^1} \to \infty, \quad t \to T_v.$$

Step 4. Conclusion. It follows from the previous steps that $u = v + \phi$ is the unique solution of (NLS) on $E_{\underline{c}}$, with initial data u_0. If $T_v < \infty$, then either $T_v < T_\infty$ or $T_\infty < \infty$. In any case,

$$\|u(t)\|_{E_{\underline{c}}} = \|v(t)\|_{H^1} + \|\phi(t)\|_{X_{\underline{c}}} \to \infty, \quad t \to T_{max},$$

which implies that u is not extendible over $E_{\underline{c}}$. □

Lemma 8.1.5. *Set $\sigma \geqslant 4$. If $u \in C([0,T], H^1(\mathbb{R}))$ is a solution of*

$$iu_t + \Delta u + \lambda |u|^\sigma u = 0 \tag{8.1.5}$$

with initial data $u_0 \in H^1(\mathbb{R})$ sufficiently small and $xu_0 \in L^2(\mathbb{R})$, then

$$\|u(t)\|_\infty \lesssim \frac{1}{t^{1/2}} \|u_0\|_2^{1/2} \|xu_0\|_2^{1/2}.$$

Proof. Consider the differential operator

$$(Pu)(t,x) = (x + 2it\partial_x)u(t,x).$$

One may easily check that

$$(Pu)(t,x) = 2ite^{i\frac{|x|^2}{4t}}\partial_x\left(e^{-i\frac{|x|^2}{4t}}u(t,x)\right)$$

and that $[P, i\partial_t + \Delta] = 0$. Applying P to (8.1.5), one obtains

$$i(Pu)_t + \Delta(Pu) + 2ite^{i\frac{|x|^2}{4t}}\partial_x\left(\left|e^{-i\frac{|x|^2}{4t}}u\right|^\sigma e^{-i\frac{|x|^2}{4t}}u\right) = 0.$$

Set $v = e^{-i\frac{|x|^2}{4t}}u$. Since

$$2ite^{i\frac{|x|^2}{4t}}\partial_x\left(|v|^\sigma v\right) = \left(\frac{\sigma}{2}+1\right)|v|^\sigma Pu + \frac{\sigma}{2}e^{i\frac{|x|^2}{2t}}|v|^{\sigma-2}v^2\overline{Pu},$$

we obtain the following Duhamel's formula for Pu:

$$(Pu)(t) = S_1(t)xu_0 + i\lambda\int_0^t S_1(t-s)\left[\left(\frac{\sigma}{2}+1\right)|v|^\sigma\overline{Pu} + \frac{\sigma}{2}e^{i\frac{|x|^2}{2t}}|v|^{\sigma-2}v^2\overline{Pu}\right](s)ds.$$

Let (q,r) and $(\gamma, \sigma+2)$ be Strichartz admissible pairs in dimension one, i.e., $r \geqslant 2$ and

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}, \quad \frac{2}{\gamma} = \frac{1}{2} - \frac{1}{\sigma+2}.$$

It then follows from Strichartz estimates that, setting

$$\mu = \frac{2\sigma(\sigma+2)}{4+\sigma} \geqslant \gamma,$$

$$\|Pu\|_{L^q((0,T);L^r)} \lesssim \|xu_0\|_2 + \|Pu\|_{L^\gamma((0,T);L^{\sigma+2})}\|u\|^{(\mu-\gamma)\sigma/\mu}_{L^\infty((0,T);H^1)}\|u\|^{\gamma\sigma/\mu}_{L^\gamma((0,T);L^{\sigma+2})}.$$

From [12, Theorem 6.2.1], if $\|u_0\|_{H^1} < \epsilon$ sufficiently small, then

$$\|u\|_{L^\infty((0,T);H^1)} + \|u\|_{L^\gamma((0,T);L^{\sigma+2})} \lesssim \epsilon.$$

Hence, adding the inequalities with $(q,r) = (\infty, 2)$ and $(q,r) = (\gamma, \sigma+2)$,

$$\|Pu\|_{L^\infty((0,T);L^2)} + \|Pu\|_{L^\gamma((0,T);L^{\sigma+2})} \lesssim \|xu_0\|_{L^2}$$

and so, for any $0 < t < T$,

$$\|\nabla v(t)\|_2 = \frac{1}{2t}\|Pu(t)\|_2 \lesssim \frac{1}{t}\|xu_0\|_2.$$

The claimed estimate now follows from Gagliardo-Nirenberg's inequality:

$$\begin{aligned}\|u(t)\|_\infty = \|v(t)\|_\infty &\lesssim \|\nabla v(t)\|_2^{1/2}\|v(t)\|_2^{1/2} \lesssim \frac{1}{t^{1/2}}\|xu_0\|_2^{1/2}\|u(t)\|_2^{1/2} \\ &\lesssim \frac{1}{t^{1/2}}\|xu_0\|_2^{1/2}\|u_0\|_2^{1/2}.\end{aligned}$$

□

Lemma 8.1.6. *Set $\sigma \geqslant 4$. Given $M > 1$, there exists $\epsilon = \epsilon(M) > 0$ such that, given $\phi_0 \in Y_{\underline{c}}$ satisfying*

$$N(\phi_0) = \sum_{n\geqslant 1} \|z(f_0)_n\|_2 + \|\partial_z (f_n)_0\|_1 < M,\ \|\phi_0\|_{X_{\underline{c}}} < \epsilon$$

the solution $\phi(t,x,y) = \sum f_n(t, x - c_n y)$ of

$$i(f_n)_t + (1+c_n^2) f_n + \lambda |f_n|^\sigma f_n = 0, \quad f_n(0) = (f_n)_0$$

is global and satisfies

$$\|\phi(t)\|_\infty \lesssim \min\left\{\epsilon, \frac{M}{t^{1/2}}\right\},\ \|\nabla\phi(t)\|_\infty \lesssim M^3,\ \|\phi(t)\|_{X_{\underline{c}}} \leqslant \frac{3}{2}\|\phi_0\|_{X_{\underline{c}}},\ t>0.$$

Proof. We write

$$\phi(t,x,y) = \sum_{n\geqslant 1} \phi_n(t,x,y), \quad \phi_n(t,x,y) = f_n(t, x - c_n y), \quad \phi_0(x,y) = \sum_{n\geqslant 1} (f_0)_n (x - c_n y).$$

Using the rescaling $h_n(t,z) = f_n(t, \sqrt{1+c_n^2} z)$, one arrives to

$$i(h_n)_t + (h_n)_{zz} + \lambda |h_n|^4 h_n = 0, \quad h_n(0,z) = (h_0)_n(z) = (f_0)_n(\sqrt{1+c_n^2} z).$$

Step 1. We now collect some properties of h_n.

First of all, it follows from [12, Theorem 6.2.1] that there exists $\epsilon_0 > 0$ such that, if $\|(h_0)_n\|_{H^1} < \epsilon_0$, then h_n is global and

$$\|h_n(t)\|_{H^1} \leqslant \frac{3}{2}\|(h_0)_n\|_{H^1},\ t > 0.$$

Using Lemma 8.1.5,

$$\|h_n(t)\|_\infty \leqslant \frac{C}{t^{1/2}} \|z(h_0)_n\|_2^{1/2} \|(h_0)_n\|_2^{1/2},\ t>0.$$

These properties imply that

$$\|h_n(t)\|_\infty \lesssim \min\left\{ \|(h_0)_n\|_{H^1}, \frac{1}{t^{1/2}} \|z(h_0)_n\|_2^{1/2} \|(h_0)_n\|_2^{1/2} \right\}.$$

Now we obtain an uniform estimate for $\|\partial_z h_n(t)\|_{L^\infty}$. Recall, from the proof of Theorem 8.1.4, that there exists $T > 0$ (independent of n) such that h_n is defined on $[0,T]$ and $\|h_n(t)\|_{H^2} \leqslant 2\|(h_n)_0\|_{H^2}$, for $0 < t < T$. Hence

$$\|\partial_z h_n(t)\|_{L^\infty} \lesssim 2\|(h_n)_0\|_{H^2},\ 0 < t < T.$$

For $t > T$, since

$$\partial_z h_n(t) = e^{it\partial_{zz}^2} \partial_z (h_0)_n + \int_0 e^{i(t-s)\partial_{zz}^2} \partial_z \left(|h_n(s)|^4 h_n(s) \right) ds,$$

one has

$$
\begin{aligned}
\|\partial_z h_n(t)\|_\infty &\lesssim \|\partial_z (h_n)_0\|_1 + \int_0^t \frac{1}{\sqrt{t-s}} \|h_n(s)\|_\infty^{\sigma-1} \|h_n(s)\|_2 \|\partial_z h_n(s)\|_2 ds \\
&\lesssim \|\partial_z (h_n)_0\|_1 + \int_0^T \frac{1}{\sqrt{t-s}} \|h_n(s)\|_{H^1}^{\sigma+1} ds \\
&\quad + \int_T^t \frac{1}{\sqrt{t-s}} \frac{1}{s^{(\sigma-1)/2}} \|z(h_0)_n\|_2^{(\sigma-1)/2} \|(h_0)_n\|_2^{(\sigma-1)/2} \|h_n(s)\|_{H^1}^2 ds \\
&\lesssim \|\partial_z (h_n)_0\|_1 + C((h_0)_n) \Big(\int_0^1 \frac{1}{\sqrt{t-s}} ds + \int_1^{t-1} \frac{1}{\sqrt{t-s}} \frac{1}{s^{(\sigma-1)/2}} ds \\
&\quad + \int_{t-1}^t \frac{1}{\sqrt{t-s}} \frac{1}{s^{(\sigma-1)/2}} ds \Big) \\
&\lesssim \|\partial_z (h_n)_0\|_1 + C((h_0)_n) \Big(\frac{1}{\sqrt{t-1}} \\
&\quad + \int_1^{t-1} \frac{1}{s^{(\sigma-1)/2}} ds + \frac{1}{(t-1)^{(\sigma-1)/2}} \int_{t-1}^t \frac{1}{\sqrt{t-s}} \Big) \\
&\lesssim \|\partial_z (h_n)_0\|_1 + C((h_0)_n) \left(\frac{1}{\sqrt{t-1}} + 1 + \frac{1}{(t-1)^{(\sigma-1)/2}} \right) \\
&\lesssim \|\partial_z (h_n)_0\|_1 + C((h_0)_n).
\end{aligned}
$$

where

$$
C((h_0)_n) = \|(h_0)_n\|_{H^1}^{\sigma+1} + \|z(h_0)_n\|_2^{(\sigma-1)/2} \|(h_0)_n\|_2^{(\sigma-1)/2} \|(h_0)_n\|_{H^1}^2 .
$$

Thus we obtain the estimate

$$
\|\partial_z h_n(t)\|_\infty \lesssim \|(h_0)_n\|_{H^2} + \|\partial_z (h_n)_0\|_1 + C((h_0)_n)
$$

Step 2. Now we write the estimates of Step 1 in terms of ϕ. Some simple computations show that, for any $1 \leqslant q \leqslant \infty$,

$$
\|h_n(t)\|_q = (1+c_n^2)^{-1/2q} \|f_n(t)\|_q, \quad \|\partial_z h_n(t)\|_q = (1+c_n^2)^{-1/2q+1/2} \|\partial_z f_n(t)\|_q
$$

$$
\|\partial_{zz} h_n(t)\|_q = (1+c_n^2)^{-1/2q+1} \|\partial_{zz} f_n(t)\|_q, \quad \|z h_n(t)\|_2 = (1+c_n^2)^{-3/4} \|z f_n(t)\|_2
$$

Notice that, if $\|\phi_0\|_{X_{\underline{c}}} < \epsilon < M$, then

$$
\|(h_n)_0\|_{H^2} \leqslant \sum_{n \geqslant 1} \|(h_n)_0\|_{H^2} \leqslant \sum_{n \geqslant 1} (1+c_n^2)^{3/4} \|(f_n)_0\|_{H^2} \leqslant \epsilon.
$$

Hence

$$
\begin{aligned}
\|\phi(t)\|_\infty &\leqslant \sum_{n \geqslant 1} \|f_n(t)\|_\infty = \sum_{n \geqslant 1} \|h_n(t)\|_\infty \\
&\lesssim \sum_{n \geqslant 1} \min \left\{ \|(h_0)_n\|_{H^1}, \frac{1}{t^{1/2}} \|z(h_0)_n\|_2^{1/2} \|(h_0)_n\|_2^{1/2} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \min\left\{\sum_{n\geqslant 1}\|(f_0)_n\|_{H^1}, \frac{1}{t^{1/2}}\sum_{n\geqslant 1}\|z(f_0)_n\|_2^{1/2}\|(f_0)_n\|_2^{1/2}\right\}\\
&\leqslant \min\left\{\epsilon, \frac{1}{t^{1/2}}\left(\sum_{n\geqslant 1}\|z(f_0)_n\|_2+\|(f_0)_n\|_2\right)\right\}\leqslant \min\left\{\epsilon, \frac{M+\epsilon}{t^{1/2}}\right\}
\end{aligned}
$$

and, in a similar fashion,

$$
\begin{aligned}
\|\nabla\phi(t)\|_\infty &\leqslant \sum_{n\geqslant 1}(1+c_n^2)^{1/2}\|\partial_z f_n(t)\|_\infty = \sum_{n\geqslant 1}\|\partial_z h_n(t)\|_\infty\\
&\lesssim \sum_{n\geqslant 1}\|(h_0)_n\|_{H^2}+\|\partial_z(h_n)_0\|_{L^1}+\|(h_0)_n\|_{H^1}^{\sigma+1}\\
&\quad+\sum_{n\geqslant 1}\|z(h_0)_n\|_2^{3/2}\|(h_0)_n\|_2^{(\sigma-1)/2}\|(h_0)_n\|_{H^1}^2\\
&\lesssim \epsilon+M+\sum_{n\geqslant 1}\|z(h_0)_n\|_2^{\sigma-1}+\|(h_0)_n\|_2^{\sigma-1}\|(h_0)_n\|_{H^1}^4\lesssim \epsilon+M+M^3\lesssim M^3.
\end{aligned}
$$

□

Theorem 8.1.7. *Set $\sigma\geqslant 4$. Given $M>1$, there exist $\epsilon(M)>0$ and $\delta=\delta(\epsilon,M)$, with $\delta(\epsilon,M)\to 0$ as $\epsilon\to 0$, such that, if $\phi_0\in Y_{\underline{c}}$ and $v_0\in H^1(\mathbb{R}^2)$ satisfy*

$$
\sum_{n\geqslant 1}\|z(f_0)_n\|_{L^2}+\|\partial_z(f_n)_0\|_{L^1}+\sum_{j\neq k}\frac{(1+c_j^2)^{1/2}\|\partial_z(f_0)_j\|_{L^2}+\|(f_0)_j\|_{L^2}}{|c_j-c_k|^{1/2}}\|(f_0)_k\|_{L^2}<M,
$$

$$
\|\phi_0\|_{X_{\underline{c}}}<\epsilon, \|v_0\|_{H^1}<\epsilon,\ \epsilon<\epsilon(M),
$$

then the solutions of (NLS) u and $\tilde{u}$, with initial data $v_0+\phi_0$ and ϕ_0, respectively, are both globally defined and satisfy

$$
\|u-\tilde{u}\|_{L^\infty((0,\infty);H^1(\mathbb{R}^2))}\leqslant \delta(\epsilon,M).
$$

Proof. We shall only prove the case $\sigma=4$, where one may develop the nonlinear term explicitly. For the general case, one uses a cut-off function to split the nonlinearity in lower and higher order terms (see the proof of Theorem 5.4.4). The main idea of the proof is, once again, to obtain a "small data global existence" result for the H^1 component (see, for example, [12, Theorem 6.2.1]).
Step 1. Setup. From the previous lemma, it follows that, for $\epsilon>0$ sufficiently small, ϕ is global and

$$
\|\phi(t)\|_{L^\infty}\lesssim \min\left\{\epsilon, \frac{M}{t^{1/2}}\right\}\lesssim M,\ \|\nabla\phi(t)\|_{L^\infty}\lesssim M^3, \|\phi(t)\|_{X_{\underline{c}}}\lesssim \epsilon,\ t>0.
$$

Fix $v_0\in H^1(\mathbb{R}^2)$ and consider the corresponding solution v of

$$
iv_t+v_{xx}+v_{yy}+\lambda(|v+\phi|^4(v+\phi)-\sum_{n\geqslant 1}|\phi_n|^4\phi_n)=0,\quad \phi_n(t,x,y)=f_n(t,x-c_ny).
$$

We recall that v is defined on $(0, T(u_0))$, where $u_0 = v_0 + \phi_0$. Since ϕ is global in $X_{\underline{c}}$, the blow-up alternative of Theorem 8.1.4 then implies that, if $T(u_0) < \infty$,

$$\|v(t)\|_{H^1} \to \infty, \ t \to T(u_0).$$

We develop the nonlinear part as

$$\lambda(|v+\phi|^4(v+\phi) - \sum_{n\geqslant 1} |\phi_n|^4\phi_n) = \sum_{i=0}^{5} g_i(v, \phi),$$

where each g_i has exactly i powers of v. Define, for $i = 3, 4, 5$, $\rho_i = i+1$ and γ_i such that (γ_i, ρ_i) is an admissible pair. In particular, $\rho_3 = \gamma_3 = 4$. Consider, for $0 < t < T(u_0)$,

$$h(t) = \|v\|_{L^\infty((0,t),H^1(\mathbb{R}^2))} + \sum_{i=3}^{5} \|v\|_{L^{\gamma_i}((0,t),W^{1,\rho_i}(\mathbb{R}^2))}. \tag{8.1.6}$$

We write Duhamel's formula,

$$v(t) = S_2(t)v_0 + \sum_{i=0}^{5} \int_0^t S_2(t-s)g_i(v(s), \phi(s))ds.$$

Therefore, for any admissible pair (q, r),

$$\|v\|_{L^q((0,t),W^{1,r}(\mathbb{R}^2))} \leqslant C\|v_0\|_{H^1} + \sum_{i=0}^{5} \left\| \int_0^\cdot S_2(\cdot - s)g_i(v(s), \phi(s))ds \right\|_{L^q((0,t),W^{1,r}(\mathbb{R}^2))}. \tag{8.1.7}$$

For the sake of simplicity, we shall omit both the temporal and spatial domains. In the next steps, we shall estimate each term of the sum by a suitable power of $h(t)$ using an appropriate Strichartz estimate. All constants depending solely on M shall be ommited.

Step 2. Estimate of higher-order terms in v on (8.1.7). Here, we shall estimate

$$\left\| \int_0^\cdot S_2(\cdot - s)g_i(v(s), \phi(s))ds \right\|_{L^q(W^{1,r})}, \quad i = 3, 4, 5.$$

Take $i = 3$. Then it follows from Step 1 that

$$\begin{aligned} \left\| \int_0^\cdot S_2(\cdot - s)g_3(v(s), \phi(s))ds \right\|_{L^q(W^{1,r})} &\lesssim \|g_3(v, \phi)\|_{L^{\gamma_3'}(W^{1,\rho_3'})} \\ &\lesssim \||v|^3|\phi|^2\|_{L^{4/3}(W^{1,4/3})} \lesssim \|v\|^3_{L^4(W^{1,4})} \\ &\lesssim \|v\|^3_{L^{\gamma_3}(W^{1,\rho_3})} \lesssim h(t)^3. \end{aligned}$$

Now we treat the case $i = 4, 5$:

$$\left\| \int_0^\cdot S_2(\cdot - s)g_i(v(s), \phi(s))ds \right\|_{L^q(W^{1,r})} \lesssim \|g_i(v, \phi)\|_{L^{\gamma_i'}(W^{1,\rho_i'})}$$

$$\lesssim \| |v|^i |\phi|^{5-i} \|_{L^{\gamma_i'}(W^{1,\rho_i'})} \lesssim \|v\|^{i-1}_{L^{\mu_i}(L^{\rho_i})} \|v\|_{L^{\gamma_i}(W^{1,\rho_i})}$$

where

$$\mu_i = \frac{(i-1)(i+1)}{2} > \gamma_i.$$

Then, through the interpolation $L^{\gamma_i} - L^{\mu_i} - L^\infty$ and the injection $H^1 \hookrightarrow L^{\rho_i}$,

$$\left\| \int_0^\cdot S_2(\cdot - s) g_i(v(s), \phi(s)) ds \right\|_{L^q(W^{1,r})} \lesssim h(t)^i, \ i = 4, 5.$$

Step 3. Estimate of the linear term in v.

$$\left\| \int_0^\cdot S_2(\cdot - s) g_1(v(s), \phi(s)) ds \right\|_{L^q(W^{1,r})} \lesssim \|g_1(v,\phi)\|_{L^1(H^1)} \lesssim \| |v| |\phi|^4 \|_{L^1(H^1)}.$$

Using the properties deduced in Step 1,

$$\begin{aligned}
\| |v| |\phi|^4 \|_{L^1(H^1)} &\lesssim \int_0^t \|\phi(s)\|^3_{L^\infty} \|\phi(s)\|_{W^{1,\infty}} \|v(s)\|_{H^1} ds \\
&\lesssim \|v\|_{L^\infty(H^1)} \|\phi\|_{L^\infty(W^{1,\infty})} \|\phi\|^{1/2}_{L^\infty(L^\infty)} \left(\int_0^t \|\phi(s)\|^{5/2}_{L^\infty} ds \right) \\
&\lesssim \|v\|_{L^\infty(H^1)} \|\phi\|_{L^\infty(W^{1,\infty})} \|\phi\|^{1/2}_{L^\infty(L^\infty)} \left(1 + \int_1^t \frac{1}{s^{5/4}} ds \right) \\
&\lesssim \epsilon^{1/2} \|v\|_{L^\infty(H^1)} \lesssim \epsilon^{1/2} h(t).
\end{aligned}$$

Step 4. Estimate of the term independent on v. Define

$$D = \{(j,k,l,m,n) \in \mathbb{N}^5 : (k,l,m,n) \neq (j,j,j,j)\}.$$

Then

$$\begin{aligned}
&\left\| \int_0^\cdot S_2(\cdot - s) g_0(v(s), \phi(s)) ds \right\|_{L^q(W^{1,r})} \lesssim \|g_0(v,\phi)\|_{L^1(H^1)} \\
&= \left\| |\phi|^4 \phi - \sum_{n \geqslant 1} |\phi_n|^4 \phi_n \right\|_{L^1(H^1)} = \left\| \sum_{(j,k,l,m,n) \notin D} \phi_j \overline{\phi_k} \phi_l \overline{\phi_m} \phi_n \right\|_{L^1(H^1)} \\
&\leqslant \sum_{(j,k,l,m,n) \notin D} \int_0^t \|(\phi_j \overline{\phi_k} \phi_l \overline{\phi_m} \phi_n)(s)\|_{H^1} ds \\
&\leqslant \int_0^t \left(\sum_{l,m,n \geqslant 1} \|\phi_l(s)\|_{L^\infty} \|\phi_m(s)\|_{L^\infty} \|\phi_n(s)\|_{L^\infty} \right) \\
&\qquad \times \left(\sum_{j \neq k} \|\nabla \phi_j(s) \phi_k(s)\|_{L^2} + \|\phi_j(s)\phi_k(s)\|_{L^2} \right) ds
\end{aligned}$$

$$\leqslant \int_0^t \left(\sum_{n\geqslant 1} \|\phi_n(s)\|_{L^\infty}\right)^3$$

$$\times \left(\sum_{j\neq k} \frac{(1+c_j^2)^{1/2}}{|c_j-c_k|^{1/2}}\|\partial_z f_j(s)\|_{L^2}\|f_k(s)\|_{L^2} + \frac{\|f_j(s)\|_{L^2}\|f_k(s)\|_{L^2}}{|c_j-c_k|^{1/2}}\right) ds.$$

Recalling that

$$\|f_k(s)\|_2 = \|(f_0)_k\|_2, \|\partial_z f_k(s)\|_2 \leqslant \frac{3}{2}\|\partial_z (f_0)_k\|_2,\ k\in\mathbb{N}, s>0,$$

one estimates

$$\sum_{j\neq k} \frac{(1+c_j^2)^{1/2}}{|c_j-c_k|^{1/2}}\|\partial_z f_j(s)\|_{L^2}\|f_k(s)\|_{L^2} + \frac{\|f_j(s)\|_{L^2}\|f_k(s)\|_{L^2}}{|c_j-c_k|^{1/2}}$$

$$\leqslant \frac{3}{2}\sum_{j\neq k} \frac{(1+c_j^2)^{1/2}}{|c_j-c_k|^{1/2}}\|\partial_z (f_0)_j\|_{L^2}\|(f_0)_k\|_{L^2} + \frac{\|(f_0)_j\|_{L^2}\|(f_0)_k\|_{L^2}}{|c_j-c_k|^{1/2}} \lesssim M.$$

Hence, by Lemma 8.1.6,

$$\left\|\int_0^\cdot S_2(\cdot - s) g_0(v(s),\phi(s))ds\right\|_{L^q(W^{1,r})} \lesssim \int_0^t \left(\sum_{n\geqslant 1}\|\phi_n(s)\|_{L^\infty}\right)^3$$

$$\lesssim \|\phi\|_{L^\infty(L^\infty)}^{1/2}\int_0^t \left(\sum_{n\geqslant 1}\|\phi_n(s)\|_{L^\infty}\right)^{5/2} \lesssim \epsilon^{1/2}\left(1+\int_1^t \frac{1}{s^{5/4}}ds\right) \lesssim \epsilon^{1/2}.$$

Step 5. Estimate of the quadratic term in v. Recalling that $\phi, \nabla\phi$ are bounded in $L^\infty(L^\infty)$, one has

$$\left\|\int_0^\cdot S_2(\cdot - s) g_2(v(s),\phi(s))ds\right\|_{L^q(W^{1,r})} \lesssim \|g_2(v,\phi)\|_{L^{4/3}(W^{1,4/3})} \lesssim \||v|^2|\phi|^3\|_{L^{4/3}(W^{1,4/3})}$$

$$\lesssim \left(\int_0^t\int |\phi|^4|v|^{8/3} + |\phi|^4|v|^{4/3}|\nabla v|^{4/3} + |\phi|^{8/3}|v|^{8/3}|\nabla\phi|^{4/3}\right)^{3/4}$$

$$\lesssim \left(\int_0^t \|\phi\|_{L^\infty}^{8/3}\left(\int |v|^2 + |\nabla v|^2 + |v|^4 + |\nabla v|^4\right)\right)^{3/4}$$

$$\lesssim \left(\left(\int_0^t \|\phi\|_{L^\infty}^{8/3}\int |v|^2+|\nabla v|^2\right) + \left(\int_0^t\int |v|^4+|\nabla v|^4\right)\right)^{3/4}$$

$$\lesssim \left(\|v\|_{L^\infty(H^1)}^2\left(1+\int_1^t \frac{1}{s^{8/6}}ds\right) + \|v\|_{L^4(W^{1,4})}^4\right)^{3/4} \lesssim h(t)^{3/2} + h(t)^3.$$

Step 6. Conclusion. Putting together Steps 2, 3, 4 and 5, there exists a constant D, depending only on M, such that

$$h(t) \leqslant D\left(\|v_0\|_{H^1} + \epsilon^{1/2} + h(t)\epsilon^{1/4} + h(t)^{3/2} + h(t)^3 + h(t)^4 + h(t)^5\right). \tag{8.1.8}$$

For ϵ sufficiently small, we arrive at

$$h(t) \lesssim \|v_0\|_{H^1} + \left(h(t)^{3/2} + h(t)^3 + h(t)^4 + h(t)^5\right)$$

The conclusion now follows from an usual bootstrap argument: If $\|v_0\|_{H^1}$ is sufficiently small, then the above inequality implies $h(t) \in [0, h_0] \cup [h_1, \infty)$, for some $\epsilon < h_0 < \delta, h_1$. Since $h(0) = \epsilon$, by continuity, one has $h(t) < \delta$, for all $t < T(u_0)$. The blow-up alternative then implies that $T(u_0) = \infty$. This implies that

$$\|u - \phi\|_{L^\infty((0,\infty),H^1(\mathbb{R}^2))} \leqslant \delta(\epsilon, M).$$

Now notice that this property is also valid for $\tilde{u}$, since it is a solution of (NLS) with $v_0 \equiv 0$. Hence

$$\|u - \tilde{u}\|_{L^\infty((0,\infty),H^1(\mathbb{R}^2))} \leqslant \|u - \phi\|_{L^\infty((0,\infty),H^1(\mathbb{R}^2))} + \|\tilde{u} - \phi\|_{L^\infty((0,\infty),H^1(\mathbb{R}^2))} \leqslant 2\delta(\epsilon, M).$$

$\square$

8.2 The plane wave transform

Definition 8.2.1 (Plane wave transform)**.** *Given $f \in C_0(\mathbb{R}^2)$, we define the plane wave transform Tf as*

$$(Tf)(x, y) = \int f(x - cy, c)dc.$$

The variables of the plane wave transform are called physical variables, while the variables of $f = f(z, c)$ are called speed variables.

Now we state some simple properties of this transform, whose proof is quite straightforward.

Lemma 8.2.2 (Algebraic properties)**.** *Fix any $f \in C_0(\mathbb{R}^2)$.*

1. *Translation property:*

$$T(f(\cdot + z_0, \cdot + c_0))(x, y) = (Tf)(x + c_0y + z_0, y);$$

2. *Scaling property:*

$$T(f(\mu\cdot, \lambda\cdot))(x, y) = \frac{1}{\lambda}(Tf)\left(\mu x, \frac{\mu}{\lambda}y\right);$$

3. *Monotonicity: if $g \in C_0(\mathbb{R}^2)$,*

$$f \leqslant g \Rightarrow Tf \leqslant Tg;$$

4. *Derivation: if $f \in C_0^1(\mathbb{R}^2)$,*

$$\nabla(Tf) = (T(f_z), -T(cf_z)).$$

Proposition 8.2.3 (L^p integrability, $p > 2$)**.** *Fix* $f \in C_0(\mathbb{R}^2)$. *Then*

$$\|Tf\|_{L^p}^2 \leqslant \int \frac{1}{|c-c'|^{2/p}} \|f(c)\|_{L_z^{p/2}} \|f(c')\|_{L_z^{p/2}} dcdc', \quad 2 < p < \infty$$

and

$$\|Tf\|_{L_y^\infty(L_x^p)} \leqslant \|f\|_{L_c^1(L_z^p)}, \quad 1 \leqslant p \leqslant \infty.$$

Consequently, T can be continuously extended in a unique way to $L_c^1(L_z^p)$, for any $1 \leqslant p < \infty$.

Proof. Take $\psi \in C_0(\mathbb{R}^2)$. Then, for $q = p/2$,

$$\begin{aligned}
&\left| \int |(Tf)(x,y)|^2 \psi(x,y) dxdy \right| \\
&\leqslant \int |f(x-cy,c) f(x-c'y,c') \psi(x,y)| dxdydcdc' \\
&\leqslant \int \left(\int |f(x-cy,c) f(x-c'y,c')|^q dxdy \right)^{1/q} \|\psi\|_{L^{q'}} dcdc' \\
&\leqslant \left(\int \frac{1}{|c-c'|^{1/q}} \left(\int |f(z,c) f(z',c')|^q dzdz' \right)^{1/q} dcdc' \right) \|\psi\|_{L^{q'}} \\
&\leqslant \left(\int \frac{1}{|c-c'|^q} \|f(c)\|_{L_z^q} \|f(c')\|_{L_z^q} dcdc' \right) \|\psi\|_{L^{q'}}.
\end{aligned}$$

This implies that

$$\|Tf\|_{L^p}^2 = \||Tf|^2\|_{L^q} \leqslant \int \frac{1}{|c-c'|^{2/p}} \|f(c)\|_{L_z^q} \|f(c')\|_{L_z^q} dcdc'.$$

On the other hand, given $\phi \in C_0(\mathbb{R})$, for a.e. $y \in \mathbb{R}$,

$$\begin{aligned}
\left| \int Tf(x,y)\phi(x) dx \right| &\leqslant \int |f(x-cy,c)||\phi(x)| dxdc \leqslant \int \left(\int |f(x-cy,c)|^p dx \right)^{1/p} \|\phi\|_{L^{p'}} dc \\
&= \|f\|_{L_c^1(L_z^p)} \|\phi\|_{L^{p'}}.
\end{aligned}$$

□

Remark 8.2.1. As a consequence of the above result, for any $f \in C_0(\mathbb{R}^2)$,

$$\|Tf\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L_c^{p'}(L_z^{p/2})}, \quad 2 < p < \infty.$$

Indeed, it follows from Young's inequality that

$$\begin{aligned}
\int \frac{1}{|c-c'|^{2/p}} \|f(c)\|_{L_z^{p/2}} \|f(c')\|_{L_z^{p/2}} dcdc' &\lesssim \left\| \frac{1}{|c-c'|^{2/p}} \|f(c)\|_{L_z^{p/2}} \right\|_{L_c^p} \|f\|_{L_c^{p'}(L_z^{p/2})} \\
&\lesssim \|f\|_{L_c^{p'}(L_z^{p/2})}^2.
\end{aligned}$$

EXAMPLE 8.2.1. It is a simple exercise to compute the transform of the characteristic function of the unit square: if $f = \mathbb{1}_{[0,1]^2}$, then

$$(Tf)(x,y) = \begin{cases} 1, & 0 \leqslant x \leqslant 1, \quad x-1 \leqslant y \leqslant x \\ x/y, & 0 \leqslant x \leqslant 1, \quad y \geqslant x \\ (x-1)/y, & 0 \leqslant x \leqslant 1, \quad y \leqslant x-1 \\ (y-x)/y, & x \leqslant 0, \qquad\quad x-1 \leqslant y \leqslant x \\ -1/y, & x \leqslant 0, \qquad\quad y \leqslant x-1 \\ (y-x+1)/y, & x \geqslant 1, \qquad\quad x-1 \leqslant y \leqslant x \\ 1/y, & x \geqslant 1, \qquad\quad y \geqslant x \\ 0, & \quad \text{otherwise} \end{cases}$$

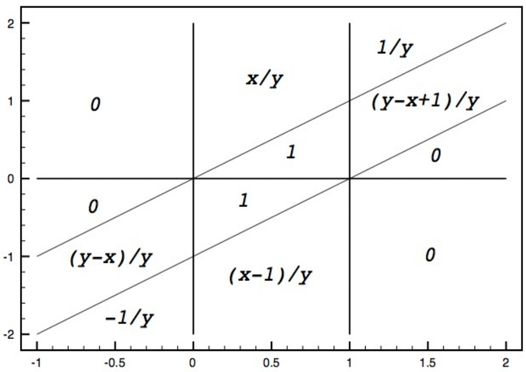

Figure 8.1: The transform of $\mathbb{1}_{[0,1]^2}$.

We claim that $Tf \notin L^2(\mathbb{R}^2)$. In fact, one has

$$\int_{\{x \geqslant 1, y \geqslant x\}} |Tf|^2 dxdy = \int_1^\infty \left(\int_1^y \frac{1}{y^2} dx \right) dy = \int_1^\infty \frac{y-1}{y^2} dy = \infty.$$

Corollary 8.2.4. *If $f \in L^1_c(L^\infty_z)$ satisfies $f \geqslant 0$ and $f \not\equiv 0$, then $Tf \notin L^2(\mathbb{R}^2)$.*

Proof. Since $f \not\equiv 0$ and $f \geqslant 0$, there exists $a > 0$ and a square $Q \subset \mathbb{R}^2$ such that $f \geqslant a\mathbb{1}_Q$, which implies that $Tf \geqslant aT\mathbb{1}_Q$. Using the translation and scaling properties, one may write the transform of $\mathbb{1}_Q$ in terms of $T\mathbb{1}_{[0,1]^2}$, which does not belong to $L^2(\mathbb{R}^2)$. $\square$

EXAMPLE 8.2.2. Consider $f = \mathbb{1}_{[0,1]^2} - \mathbb{1}_{[0,1]\times[1,2]}$. Then, using the expression for $T\mathbb{1}_{[0,1]^2}$ and the translation property, it is easy to check that $Tf \in L^2(\mathbb{R}^2)$.

EXAMPLE 8.2.3. Let's compute the transform of $f(z,c) = e^{-c^2-z^2}$:

$$\int e^{-c^2-(x-cy)^2} dc = \int e^{-c^2-x^2+2cxy-c^2y^2} dc = e^{-x^2} \int e^{-(1+y^2)\left(c - \frac{xy}{1+y^2}\right)^2} e^{\frac{x^2y^2}{1+y^2}} dc$$
$$= e^{-\frac{x^2}{1+y^2}} \frac{1}{\sqrt{1+y^2}} \int e^{-c^2} dc = 2\pi \frac{1}{\sqrt{4\pi(1+y^2)}} e^{-\frac{x^2}{1+y^2}}.$$

It is interesting that f is the kernel of the 2D-heat kernel in Fourier variables at time $t = 1$, and that its transform is the 1D-heat kernel in physical variables at time $t = 1+y^2$.

Proposition 8.2.5. *Given $f \in L^1(\mathbb{R}^2)$, one has*

$$(Tf)(x,y) = \mathcal{F}_\xi^{-1}\left[(\mathcal{F}_{z,c}f)(\cdot, y\cdot)\right](x), \quad a.e.\ (x,y) \in \mathbb{R}^2$$

where $\mathcal{F}_\xi$ denotes the Fourier transform in the ξ variable. Consequently, one has the following inversion formula

$$f(z,c) = \mathcal{F}_{\xi,\eta}^{-1}\left[(\mathcal{F}_x(Tf))\left(\xi, \frac{\eta}{\xi}\right)\right](z,c), \quad a.e.\ (z,c) \in \mathbb{R}^2$$

Proof. First of all, since $f \in L^1(\mathbb{R}^2)$, one has $Tf \in L^\infty_y(L^1_x)$. For a.e. $y \in \mathbb{R}$, one may then take the Fourier transform of $Tf(\cdot, y)$:

$$\begin{aligned}\mathcal{F}_x[Tf(\cdot,y)](\xi) &= \int f(x-cy,c)e^{-2\pi i x\xi}dxdc = \int f(z,c)e^{-2\pi i z\xi - 2\pi i c y\xi}dzdc \\ &= (\mathcal{F}_{z,c}f)(\xi, y\xi).\end{aligned}$$

□

Proposition 8.2.6. *Take $f \in L^1_c(L^2_z) \cap \mathcal{F}(H^1(\mathbb{R}^2))$. For each $y \in \mathbb{R}$, define $\Gamma_y = \{(\xi, y\xi) : \xi \in \mathbb{R}\}$ and let*

$$\pi_y : H^1(\mathbb{R}^2) \mapsto L^2(\Gamma_y)$$

be the usual trace operator on Γ_y. Moreover, consider the isomorphism

$$j : L^2(\Gamma_y) \mapsto L^2(\mathbb{R})$$

$$j(f)(\xi) = f(\xi, y\xi) \quad \xi \in \mathbb{R}.$$

and $\Pi_y = j \circ \pi_y$. Then, for a.e. $y \in \mathbb{R}$,

$$\mathcal{F}_x[Tf(\cdot,y)] = \Pi_y \mathcal{F}_{z,c} f.$$

Proof. Take $f_n \to f$ in $L^1_c(L^2_z) \cap \mathcal{F}(H^1(\mathbb{R}^2))$, $f_n \in S(\mathbb{R}^2)$. In particular,

$$Tf_n \to Tf \text{ in } L^\infty_y(L^2_x).$$

Hence, for a.e $y \in \mathbb{R}^2$, using the previous result,

$$\mathcal{F}_x[Tf(\cdot,y)] = \lim \mathcal{F}_x[Tf_n(\cdot,y)] = \lim \Pi_y\mathcal{F}_{z,c}f_n = \Pi_y\mathcal{F}_{z,c}f.$$

□

Corollary 8.2.7 (Transform of the product of two functions)**.** *For $f, g \in C_0(\mathbb{R}^2)$,*

$$T(fg)(x,y) = \int_{\mathbb{R}} \left(T\left(e^{-2\pi i k c}f\right)(x,y)\right)\left(T\left(e^{2\pi i k c}g\right)(x,y)\right)dk.$$

Proof. The result is a simple application of Proposition 8.2.5. First of all, notice that

$$\begin{aligned}(\mathcal{F}_{z,c}f * \mathcal{F}_{z,c}g)(\xi, y\xi) &= \int (\mathcal{F}_{z,c}f)(l,k')(\mathcal{F}_{z,c}g)(\xi-l, y\xi-k')dldk' \\ &= \int (\mathcal{F}_{z,c}f)(l,yl+k)(\mathcal{F}_{z,c}g)(\xi-l, y(\xi-l)-k)dldk \\ &= \int (\mathcal{F}_{z,c}e^{-2\pi ick}f)(l,yl)(\mathcal{F}_{z,c}e^{2\pi ick}g)(\xi-l, y(\xi-l))dldk \\ &= \int \left((\mathcal{F}_{z,c}e^{-2\pi ick}f)(\cdot, y\cdot) * (\mathcal{F}_{z,c}e^{2\pi ick}g)(\cdot, y\cdot)\right)(\xi)dk.\end{aligned}$$

Hence

$$\begin{aligned}T(fg)(\cdot,y) &= \mathcal{F}_\xi^{-1}\left((\mathcal{F}_{z,c}f * \mathcal{F}_{z,c}g)(\xi, y\xi)\right) \\ &= \mathcal{F}_\xi^{-1}\left(\int \left((\mathcal{F}_{z,c}e^{-2\pi ick}f)(\cdot, y\cdot) * (\mathcal{F}_{z,c}e^{2\pi ick}g)(\cdot, y\cdot)\right)(\xi)dk\right) \\ &= \int \mathcal{F}_\xi^{-1}\left((\mathcal{F}_{z,c}e^{-2\pi ick}f)(\cdot, y\cdot)\right)\mathcal{F}_\xi^{-1}\left((\mathcal{F}_{z,c}e^{2\pi ick}g)(\cdot, y\cdot)\right)dk \\ &= \int T\left(e^{-2\pi ick}f\right)(\cdot,y)T\left(e^{2\pi ick}g\right)(\cdot,y)dk.\end{aligned}$$

□

Proposition 8.2.8 (Parseval's identity for the plane wave transform)**.** *For any $p \geqslant 1$, if $f \in L_c^1(L_z^p)$ and $g \in L_c^1(L_z^{p'})$, one has*

$$\int (Tf)(x,y)g(x,y)dxdy = \int f(z,c)(Tg)(z,-c)dzdc.$$

Proof. If $f, g \in C_0(\mathbb{R}^2)$, the result follows from

$$\begin{aligned}\int (Tf)(x,y)g(x,y)dxdy = \int f(x-cy,c)g(x,y)dxdydc &= \int f(z,c)g(z+cy,y)dydcdz \\ &= \int f(z,c)(Tg)(z,-c)dzdc.\end{aligned}$$

The general case is obtained by a density argument. □

Corollary 8.2.9. *If $f \in L_c^1(L_z^2)$ is such that $Tf \equiv 0$, then $f \equiv 0$.*

Proof. Fix $h \in \mathcal{S}(\mathbb{R}^2)$, $h = h(z,c)$. Let $\psi \in C^\infty(\mathbb{R})$ be such that $\psi \equiv 1$ in $[-1,1]$ and $\psi \equiv 0$ in $\mathbb{R}\backslash[-2,2]$. For any $\epsilon > 0$, define

$$\psi_\epsilon(\xi) = \psi\left(\frac{\xi}{\epsilon}\right), \quad h_\epsilon = h - \left(\mathcal{F}_\xi^{-1}\psi_\epsilon\right) \star_z h.$$

It is clear that $\mathcal{F}_z h_\epsilon \in S(\mathbb{R}^2)$ has support outside the strip $\{|\xi| < \epsilon\}$. Furthermore,

$$\|h - h_\epsilon\|_{L_c^\infty(L_z^2)} = \|\psi_\epsilon \mathcal{F}_z h\|_{L_c^\infty(L_z^2)} \to 0, \quad \epsilon \to 0.$$

Setting

$$g_\epsilon(\xi,\eta) := (\mathcal{F}_z h_\epsilon)\left(\xi, \frac{\eta}{\xi}\right),$$

it follows that $g_\epsilon \in \mathcal{S}(\mathbb{R}^2)$ and so $\mathcal{F}^{-1}_{\xi,\eta} g_\epsilon \in \mathcal{S}(\mathbb{R}^2)$. By Proposition 8.2.5, $T(\mathcal{F}^{-1}_{\xi,\eta} g_\epsilon) = h_\epsilon$. Using Proposition 8.2.8,

$$\begin{aligned}\int f(z,c)h(z,c)dzdc = \lim \int f(z,c)h_\epsilon(z,c)dzdc = \lim \int f(z,c)\left(T(\mathcal{F}^{-1}_{\xi,\eta} g_\epsilon)\right)(z,c)dzdc \\ = \lim \int Tf(z,-c)(\mathcal{F}^{-1}_{\xi,\eta} g_\epsilon)(z,c)dzdc = 0\end{aligned}$$

and so $f \equiv 0$. □

Corollary 8.2.10 (L^2 integrability)**.** *Take $f \in L^1(\mathbb{R}^2)$. Then*

$$\|Tf\|^2_{L^2(\mathbb{R}^2)} = \int \frac{1}{|\xi|} |\mathcal{F}_z f|^2 (\xi,c) d\xi dc = \int \frac{1}{|\xi|} |\mathcal{F}_{z,c} f|^2 (\xi,\eta) d\xi d\eta = \|f\|_{L^2_c(\dot{H}^{1/2}_z)}.$$

Consequently, if the spectrum of f lies in $(] - M, -\epsilon[\cup]\epsilon, M[) \times] - M, M[$, for some $M, \epsilon > 0$, $Tf \in L^2(\mathbb{R}^2)$.

Proof. This is a direct consequence of the formula for T in terms of the Fourier transform. □

Corollary 8.2.11. *Take $f \in L^1(\mathbb{R}^2)$ and $g \in L^1(\mathbb{R})$. Then*

$$T(f(\cdot,c) * g) = (Tf)(\cdot,y) * g.$$

Proof. By definition,

$$\begin{aligned}T(f(\cdot,c) * g)(x,y) = \int (f(\cdot,c) * g)(x - cy)dc = \int f(x - cy - z, c)g(z)dzdc \\ = ((Tf)(\cdot,y) * g)(x).\end{aligned}$$

□

Corollary 8.2.12 (Convolution through the plane wave transform)**.** *Take $f_1, f_2 \in L^1(\mathbb{R})$. If $f = f_1 \otimes f_2$, that is, $f(z,c) = f_1(z)f_2(c)$, then*

$$(Tf)(x,1) = (f_1 * f_2)(x).$$

More generally, for any $y \neq 0$, one has

$$(Tf)(x,y) = (f_1 * \Theta_y f_2)(x), \quad \Theta_y f_2(\cdot) = \frac{1}{|y|} f_2(\cdot/y).$$

and, for $y = 0$, $(Tf)(x,0) = \left(\int f_2(c)\right) f_1(x)$.

Proof. The result is trivial for $y = 0$. Given $y \neq 0$, one has

$$\begin{aligned}(Tf)(x,y) &= \int f(x-cy,c)dc = \int f_1(x-cy)f_2(c)dc = \int \frac{1}{|y|} f_1(x-c')f_2\left(\frac{c'}{y}\right)dc' \\ &= (f_1 * \Theta_y f_2)(x).\end{aligned}$$

□

REMARK 8.2.2. Corollary 8.2.12 and Proposition 8.2.3 give a new proof of Young's inequality:

$$\|f_1 * f_2\|_{L^p} \leqslant \|Tf\|_{L^\infty_y(L^p_x)} \leqslant \|f\|_{L^1_c(L^p_z)} = \|f_1\|_{L^p}\|f_2\|_{L^1}.$$

Moreover, since, for a.e. $y \in \mathbb{R}$,

$$\begin{aligned}\left|\int (Tf(x,y) - Tf(x,0))\psi(x)dx\right| &= \left|\int (f(x,c) - f(x-cy,c))\psi(x)dcdx\right| \\ &\leqslant \int |f_2(c)| \|f_1(\cdot) - f_1(\cdot - cy)\|_{L^p} \|\psi\|_{L^{p'}} dc,\end{aligned}$$

when $y \to 0$, by Dominated Convergence Theorem,

$$\|Tf(x,y) - Tf(x,0)\|_{L^p_x} \leqslant \int |f_2(c)| \|f_1(\cdot) - f_1(\cdot - cy)\|_{L^p} dc \to 0$$

which is a new way to prove convergence of mollifiers.

8.3 Solving some linear PDE's using the plane wave transform

In this short section, we apply the plane wave transform to solve some classical equations.

EXAMPLE 8.3.1 (The wave equation). Consider the wave equation in $\mathbb{R}^2$

$$u_{tt} - u_{xx} - u_{yy} = 0.$$

If one supposes that $u(x,y,t) = f(t, x-cy)$, one arrives to

$$f_{tt} - (1+c^2)f_{zz} = 0,$$

which may be explicitly solved:

$$f(t,z,c) = \frac{f_0(z-\sqrt{1+c^2}t,c) + f_0(z+\sqrt{1+c^2}t,c)}{2} + \frac{1}{2\sqrt{1+c^2}}\int_{z-\sqrt{1+c^2}t}^{z+\sqrt{1+c^2}t} f_1(s,c)ds,$$

Notice that we introduced on purpose the speed c as an independent variable of f. Now, taking the transform of $f(t,\cdot)$,

$$(Tf)(t,x,y) = \int \frac{f_0(x-cy-\sqrt{1+c^2}t,c) + f_0(x-cy+\sqrt{1+c^2}t,c)}{2}$$

$$+\frac{1}{2\sqrt{1+c^2}}\left(\int_{x-cy-\sqrt{1+c^2}t}^{x-cy+\sqrt{1+c^2}t} f_1(s,c)ds\right)dc,$$

one obtains a solution of the 2D wave equation. We remark that, unlike Poisson's formula for classical solutions, this representation does not involve derivatives of f_1.

EXAMPLE 8.3.2 (The Schrödinger equation). Take the linear Schrödinger equation:

$$iu_t + u_{xx} + u_{yy} = 0.$$

Introducing the plane wave ansatz $u(x,y,t) = f(t, x-cy)$,

$$if_t + (1+c^2)f_{zz} = 0.$$

The solution is given by

$$f(t,z,c) = \frac{1}{\sqrt{4i\pi(1+c^2)t}}\int e^{\frac{i|z-w|^2}{4(1+c^2)t}} f_0(w,c)dw$$

which, through the plane wave transform, gives a family of solutions to the two dimensional Schrödinger equation.

$$u(t,x,y) = \int \frac{1}{\sqrt{4i\pi(1+c^2)t}} e^{\frac{i|x-cy-w|^2}{4(1+c^2)t}} f_0(w,c)dwdc.$$

EXAMPLE 8.3.3 (The heat equation). We now consider a slightly different application of the plane wave transform. Take the heat equation in one dimension:

$$u_t = u_{xx}.$$

Here, instead of taking a plane wave in two spatial dimensions, we consider a plane wave in time and space. The ansatz $u(t,x) = f(x-ct)$ implies that

$$-cf' = f'',\ i.e.,\ f(z,c) = A(c)e^{-cz} + B(c).$$

Setting $B(c) = 0$ and applying the transform T, one arrives to the following family of solutions to the heat equation:

$$u(t,x) = \int A(c)e^{-cx+c^2t}dc.$$

Actually, if one is given an initial condition $u(0,x) = u_0(x)$, then one observes that A is the inverse Laplace transform of u_0. Although this integral representation is certainly valid, the presence of the term e^{-cx} implies a strong decay of A at infinity. Now, *if one replaces c with ic in the integral representation, one still obtains a solution to the heat equation.* This procedure makes no sense when one starts with the ansatz

$u(t,x) = f(x-ct)$; however, the integral representation is still meaningful for complex values of c. Hence another family of solutions is

$$u(t,x) = \int \tilde{A}(c) e^{-icx - c^2 t} dc.$$

In this situation, one sees that $\tilde{A}$ is none other than the inverse Fourier transform of u_0 and so

$$u(t,x) = \mathcal{F}_c \left(e^{-c^2 t} \mathcal{F}_x^{-1} u_0 \right),$$

which is precisely the solution of the heat equation given by the Fourier transform. This integral representation was studied in [17] as a generalization of the Fourier transform.

Example 8.3.4 (The Schrödinger equation II). As in the previous example, one may derive a family of solutions to the 1D-Schrödinger equation using T in both space and time:

$$u(t,x) = \int A(c) e^{-icx + ic^2 t} dc.$$

Givan an initial condition u_0, A corresponds to the Fourier transform of u_0. The substitution $c \mapsto -ic$ gives the family

$$u(t,x) = \int \tilde{A}(c) e^{-cx - ic^2 t} dc$$

which is connected to the Laplace transform.

We finish with a result stating that the above classical arguments to reduce the dimension of the equations are valid in a functional setting. The result, although stated only for the linear Schrödinger equation, may also be extended to more equations.

Proposition 8.3.1. *Let S_1 and S_2 be the free Schrödinger groups in dimensions one and two, respectively. Given $f \in L^1_c(L^2_z)$, one has*

$$T(S_1((1+c^2)t)f(c)) = S_2(t)Tf \text{ in } \mathcal{S}'(\mathbb{R}^2).$$

Proof. Suppose that $f \in \mathcal{S}(\mathbb{R}^2)$. Then, from the formula of Proposition 8.2.5,

$$\begin{aligned}
&\mathcal{F}_x \left[S_2(t)Tf \right](\xi, y) = \mathcal{F}_x \left[\mathcal{F}^{-1}_{\xi,\eta} \left(e^{-4\pi^2 i(\xi^2+\eta^2)t} \mathcal{F}_{x,y} Tf \right) \right](\xi, y) \\
&= \mathcal{F}^{-1}_{\eta} \left[e^{-4\pi^2 i(\xi^2+\eta^2)t} \mathcal{F}_{x,y} Tf \right](\xi, y) = \mathcal{F}^{-1}_{\eta} \left[e^{-4\pi^2 i(\xi^2+\eta^2)t} \mathcal{F}_y \left((\mathcal{F}_{z,c} f)(\xi, y\xi) \right) \right](\xi, y) \\
&= \mathcal{F}^{-1}_{\eta} \left[e^{-4\pi^2 i(\xi^2+\eta^2)t} \frac{1}{|\xi|} \left((\mathcal{F}_z f)(\xi, -\eta/\xi) \right) \right](\xi, y) \\
&= \mathcal{F}_{\eta} \left[e^{-4\pi^2 i(\xi^2+\eta^2)t} \frac{1}{|\xi|} \left((\mathcal{F}_z f)(\xi, \eta/\xi) \right) \right](\xi, y) \\
&= \mathcal{F}_{\eta} \left[e^{-4\pi^2 i(1+\eta^2)\xi^2 t} \left((\mathcal{F}_z f)(\xi, \eta) \right) \right](\xi, y\xi) \\
&= \mathcal{F}_{z,\eta} \left[\mathcal{F}^{-1}_{\xi} e^{-4\pi^2 i(1+\eta^2)\xi^2 t} \left((\mathcal{F}_z f)(\xi, \eta) \right) \right](\xi, y\xi) \\
&= \mathcal{F}_x T(S_1((1+c^2)t)f(c))(\xi, y).
\end{aligned}$$

The general case follows by a density argument. $\square$

8.4 The continuous case

Consider

$$X = \left\{\phi \in L^\infty(\mathbb{R}^2) : \phi = Tf,\ (1+|c|)f \in L^1_c(H^1_z) \cap L^\infty_c(H^1_z)\right\}$$

endowed with the induced norm

$$\|\phi\|_X = \|(1+|c|)f\|_{L^1_c(H^1_z)} + \|(1+|c|)f\|_{L^\infty_c(H^1_z)}.$$

Notice that such a norm is well-defined, by Corollary 8.2.9. Furthermore, since $L^1_c(H^1_z) \hookrightarrow L^1_c(L^\infty_z)$, one has, by Proposition 8.2.3 and Remark 8.2.1,

$$\|\phi\|_{W^{1,4}}, \|\phi\|_{L^\infty} \lesssim \|\phi\|_X.$$

Now consider the subspace of $L^4(\mathbb{R}^2)$

$$E = H^1(\mathbb{R}^2) + X,$$

endowed with the semi-norm

$$\|u\|_E = \inf_{u=v+\phi} \left\{\|v\|_{H^1} + \|\phi\|_X\right\}.$$

Lemma 8.4.1. *The semi-norm $\|\cdot\|_E$ is a norm in E. Moreover, $(E, \|\cdot\|_E)$ is a Banach space and $E \hookrightarrow L^4(\mathbb{R}^2)$.*

Proof. This is a classical result (see, for example, [7, Lemma 2.3.1]) □

Lemma 8.4.2. *Let $\{S_2(t)\}_{t\in\mathbb{R}}$ be the Schrödinger group in dimension two. Then*

$$\|S_2(t)\|_{E\to E} = 1,\ t \in \mathbb{R}.$$

Proof. Given $u \in E$, write $u = v + \phi$, $v \in H^1(\mathbb{R}^2)$, $\phi \in X$, $\phi = Tf$. Then, by Proposition 8.3.1,

$$\begin{aligned}\|S_2(t)\phi\|_X &= \|S_1((1+c^2)t)f\|_{L^1_c(H^1_z)} + \|S_1((1+c^2)t)f\|_{L^\infty_c(L^2_z)} \\ &= \|f\|_{L^1_c(H^1_z)} + \|f\|_{L^\infty_c(L^2_z)} = \|\phi\|_X.\end{aligned}$$

Hence

$$\|S_2(t)v\|_{H^1} + \|S_2(t)\phi\|_X = \|v\|_{H^1} + \|\phi\|_X$$

and, taking the infimum on both sides, one concludes the proof. □

For the sake of clarity, we define do we mean as a solution of (NLS) over E.

Definition 8.4.3. *Given $u_0 \in E$ and $u \in C([0,T], E)$, $T > 0$, we say that u is a solution of (NLS) with initial data u_0 if u satisfies the Duhamel formula*

$$u(t) = S_2(t)u_0 - i\lambda \int_0^t S_2(t-s)|u(s)|^\sigma u(s)ds,\ 0 \leqslant t \leqslant T,$$

where S_2 is the Schrödinger group in dimension two.

Lemma 8.4.4. *For any $\sigma \geqslant 1$ and $u_0 \in E$, if $u_1, u_2 \in C([0,T], E)$, $T > 0$, are two solutions of (NLS) with initial data u_0, then $u_1 \equiv u_2$.*

Proof. Since u_1, u_2 are two solutions with the same initial data,

$$(u_1 - u_2)(t) = i\lambda \int_0^t S_2(t-s)\left(|u_1(s)|^\sigma u_1(s) - |u_2(s)|^\sigma u_2(s)\right) ds,\ 0 \leqslant t \leqslant T. \tag{8.4.1}$$

Let $r \geqslant 2$ be such that $r' \geqslant 4/(\sigma+1)$. Take p such that

$$r'p \geqslant 4, \quad r'p' \geqslant 4/\sigma.$$

Then, since

$$||u_1|^\sigma u_1 - |u_2|^\sigma u_2| \leqslant C(|u_1|^\sigma + |u_2|^\sigma)|u_1 - u_2|,\ 0 \leqslant s \leqslant T,$$

using Hölder's inequality,

$$\||u_1|^\sigma u_1 - |u_2|^\sigma u_2|\|_{L^{r'}} \leqslant C(\|u_1\|^\sigma_{L^{r'p'\sigma}} + \|u_2\|^\sigma_{L^{r'p'\sigma}})\|u_1 - u_2\|_{L^{r'p}}.$$

Define γ and q such that $(\gamma, r'p)$ and (q, r) are admissible pairs, i.e.,

$$\frac{2}{\gamma} = \frac{1}{2} - \frac{1}{r'p}, \quad \frac{2}{q} = \frac{1}{2} - \frac{1}{r}.$$

For any $J = [0,t]$, $0 < t \leqslant T$ and for $q = 2r/(r-2)$, it follows that

$$\||u_1|^\sigma u_1 - |u_2|^\sigma u_2|\|_{L^{q'}(J,L^{r'})} \leqslant C(\|u_1\|^\sigma_{L^\infty(J,L^{r'p'\sigma})} + \|u_2\|^\sigma_{L^\infty(J,L^{r'p'\sigma})})\|u_1 - u_2\|_{L^{q'}(J,L^{r'p})}$$

Using Strichartz's estimates in (8.4.1),

$$\|u_1 - u_2\|_{L^\gamma(J,L^{r'p})} \leqslant C(\|u_1\|^\sigma_{L^\infty(J,L^{r'p'\sigma})} + \|u_2\|^\sigma_{L^\infty(J,L^{r'p'\sigma})})\|u_1 - u_2\|_{L^{q'}(J,L^{r'p})}.$$

Since $E \hookrightarrow L^{r'p'\sigma}(\mathbb{R}^2)$,

$$\|u_1 - u_2\|_{L^\gamma(J,L^{r'p})} \leqslant C(\|u_1\|^\sigma_{L^\infty(J,E)} + \|u_2\|^\sigma_{L^\infty(J,E)})\|u_1 - u_2\|_{L^{q'}(J,L^{r'p})} \leqslant C'\|u_1 - u_2\|_{L^{q'}(J,L^{r'p})}$$

where C' does not depend on J. The result now follows from [12, Lemma 4.2.2]. □

Lemma 8.4.5. *For any $\sigma \geqslant 1$ and $u \in E$, $|u|^\sigma u \in H^{-1}(\mathbb{R}^2)$.*

Proof. Writing $u = v + \phi$,

$$|u|^{\sigma+1} \lesssim |v|^{\sigma+1} + |\phi|^{\sigma+1}.$$

Since $v \in H^1(\mathbb{R}^2)$, $v \in L^{\sigma+2}(\mathbb{R}^2)$ and so $|v|^{\sigma+1} \in L^{\frac{\sigma+2}{\sigma+1}}(\mathbb{R}^2)$. On the other hand, recalling that $\phi \in L^p$, for any $p \geqslant 4$, $|\phi|^{\sigma+1} \in L^2(\mathbb{R}^2)$. Hence $|u|^{\sigma+1} \in L^2(\mathbb{R}^2) + L^{\frac{\sigma+2}{\sigma+1}}(\mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}^2)$. □

Theorem 8.4.6. *Fix $\sigma \geqslant 1$ and $u_0 \in E$. Then there exists a unique maximal solution $u \in C([0,T),E)$ of (NLS) (cf. definition 8.4.3) such that $u(0) = u_0$. The solution depends continuously on the initial data. Furthermore, if $T < \infty$, then*

$$\|u(t)\|_E \to \infty, \quad t \to T.$$

Proof of Theorem 8.4.6. Write $u_0 = v_0 + \phi_0$, for some $v_0 \in H^1(\mathbb{R}^2)$, $\phi_0 \in X$. Once again, define r, p, q and γ as in the proof of Lemma 8.4.4. In what follows, constants involving ϕ_0 shall be ommited from the estimates. Given $T, M > 0$, consider the space

$$\mathcal{E} = \Big\{ u = v + S_2\phi_0 : v \in L^\infty((0,T);H^1(\mathbb{R}^2)) \cap L^\gamma((0,T);W^{1,r'p}(\mathbb{R}^2)),$$
$$\|v\|_{L^\infty((0,T);H^1(\mathbb{R}^2))} < M,\ \|v\|_{L^\gamma((0,T);W^{1,r'p}(\mathbb{R}^2))} < M \Big\}.$$

endowed with the distance

$$d(u_1,u_2) = \|u_1 - u_2\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} + \|u_1 - u_2\|_{L^\gamma((0,T);L^{r'p}(\mathbb{R}^2))}.$$

As in the proof of Theorem 8.1.4, $\mathcal{E}$ is a complete metric space. Define, for $u \in \mathcal{E}$,

$$\Phi(u) = S_2(t)\phi_0 + \left(S_2(t)v_0 + \int_0^t S_2(t-s)|u(s)|^\sigma u(s)ds \right). \tag{8.4.2}$$

One has

$$\begin{aligned}
\||u|^\sigma u\|_{W^{1,r'}} \lesssim& \|u\|^\sigma_{L^{r'p'\sigma}} \|u\|_{W^{1,r'p}} \\
\lesssim& (\|u - S_2\phi_0\|_{L^{r'p'\sigma}} + \|S_2\phi_0\|_{L^{r'p'\sigma}})^\sigma (\|u - S_2\phi_0\|_{W^{1,r'p}} + \|S_2\phi_0\|_{W^{1,r'p}}) \\
\lesssim& (\|u - S_2\phi_0\|_{H^1} + \|\phi_0\|_X)^\sigma (\|u - S_2\phi_0\|_{W^{1,r'p}} + \|\phi_0\|_X),
\end{aligned}$$

where the last estimate follows from the fact that $X \hookrightarrow W^{1,4}(\mathbb{R}^2)$ and that S_2 is a unitary group over X. Hence

$$\begin{aligned}
&\||u|^\sigma u\|_{L^{q'}((0,T);W^{1,r'})} \\
\lesssim& (\|u - S_2\phi_0\|_{L^\infty((0,T);H^1)} + \|\phi_0\|_X)^\sigma (\|u - S_2\phi_0\|_{L^{q'}((0,T);W^{1,r'p})} + T^{\frac{1}{q'}}\|\phi_0\|_X) \\
\lesssim& (1+M)^\sigma (T^{\frac{\gamma - q'}{\gamma q'}} M + T^{\frac{1}{q'}}),
\end{aligned}$$

It follows from Strichartz estimates that

$$\begin{aligned}
\|\Phi(u) - S_2\phi_0\|_{L^\infty((0,T),H^1(\mathbb{R}^2))} &+ \|\Phi(u) - S_2\phi_0\|_{L^q((0,T),W^{1,r}(\mathbb{R}^2))} \\
&\lesssim \|v_0\|_{H^1} + (1+M)^\sigma (T^{\frac{\gamma - q'}{\gamma q'}} M + T^{\frac{1}{q'}}).
\end{aligned} \tag{8.4.3}$$

Now we show a Lipschitz estimate for Φ: since

$$||u_1|^\sigma u_1 - |u_2|^\sigma u_2| \lesssim (|u_1|^\sigma + |u_2|^\sigma)|u_1 - u_2|,\ 0 \leqslant s \leqslant T,$$

using Hölder's inequality,

$$\||u_1|^\sigma u_1 - |u_2|^\sigma u_2\|_{L^{r'}} \lesssim (\|u_1\|^\sigma_{L^{r'p'\sigma}} + \|u_2\|^\sigma_{L^{r'p'\sigma}})\|u_1 - u_2\|_{L^{r'p}},\ 0 \leqslant s \leqslant T,$$

It follows that

$$\begin{aligned}&\||u_1|^\sigma u_1 - |u_2|^\sigma u_2\|_{L^{q'}((0,T);L^{r'})}\\ &\leqslant C(\|u_1\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})} + \|u_2\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})})\|u_1 - u_2\|_{L^{q'}((0,T);L^{r'p})}\end{aligned}$$

Using Strichartz estimates in (8.4.2),

$$\begin{aligned} d(\Phi(u_1), \Phi(u_2)) &\lesssim (\|u_1\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})} + \|u_2\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})})\|u_1 - u_2\|_{L^{q'}((0,T);L^{r'p})}\\ &\lesssim (\|u_1 - S_2(t)\phi_0\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})} + \|u_2 - S_2(t)\phi_0\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})} \qquad (8.4.4)\\ &+ \|S_2(t)\phi_0\|^\sigma_{L^\infty((0,T);L^{r'p'\sigma})})\|u_1 - u_2\|_{L^{q'}((0,T);L^{r'p})}\\ &\lesssim (1 + M^\sigma)T^{\frac{\gamma-q'}{\gamma q'}} d(u_1, u_2)\end{aligned}$$

Hence, for M fixed (but sufficiently large) and $T > 0$ small, (8.4.3) ensures that $\Phi : \mathcal{E} \to \mathcal{E}$ and (8.4.4) implies that Φ is a strict contraction over $\mathcal{E}$. Therefore, by Banach's fixed point theorem, there exists a local solution u with initial data u_0, which, by Lemma 8.4.4, may be extended to a maximal interval of existence $[0, T_{max})$. Moreover, if $T_{max} < \infty$, one has

$$\lim_{t\to T_{max}} \|u(t) - S_2(t)\phi\|_{H^1} = \infty,\ \phi \in X.$$

□

Remark 8.4.1. The existence part of Theorem 8.4.6 could be proved as in the numerable case, by performing a fixed-point argument only on the H^1 component. We see, however, that, in the continuous case, the integrability properties of the X component allows a more direct argument. Moreover, notice that, when one applies the fixed-point theorem, one only ensures the uniqueness of a solution of the form $v(t) + S_2(t)\phi_0$ and not the uniqueness of u (which is proved using Lemma 8.4.4).

Theorem 8.4.7. *Set $\sigma \geqslant 3$. Given $M > 1$, there exist $\epsilon(M) > 0$ and $\delta = \delta(\epsilon, M)$, with $\delta(\epsilon, M) \to 0$ as $\epsilon \to 0$, such that, if $\phi_0 = Tf_0 \in X$ and $v_0 \in H^1(\mathbb{R}^2)$ satisfy*

$$\left\|\frac{f_0}{(1+c^2)^{1/2}}\right\|_{L^1_c(L^1_z)} + \|(f_0)_z\|_{L^1_c(L^1_z)} + \|f_0\|_{L^1_c(H^2_z)} + \|cf_0\|_{L^1_c(H^2_z)} < M,$$

$$\|\phi_0\|_X < \epsilon, \|v_0\|_{H^1} < \epsilon,\ \epsilon < \epsilon(M),$$

then the solution u of (NLS) with initial data $v_0 + \phi_0$ is globally defined. Moreover, if S_2 is the free Schrödinger group in dimension two,

$$\|u - S_2\phi_0\|_{L^\infty((0,\infty),H^1(\mathbb{R}^2))} \leqslant \delta(\epsilon, M)$$

Sketch of the proof of Theorem 8.4.7. The proof is very similar to that of the numerable case and again will only be done for $\sigma = 4$. We start with some estimates for $S_2\phi_0$: using Propositions 8.2.3 and 8.3.1,

$$\begin{aligned}\|S_2(t)Tf_0\|_{L^\infty} &= \|T(S_1((1+c^2)t)f_0)\|_{L^\infty} \leqslant \|S_1((1+c^2)t)f_0\|_{L^1_c(L^\infty_z)} \\ &\lesssim \frac{1}{t^{1/2}}\left\|\frac{f_0}{(1+c^2)^{1/2}}\right\|_{L^1_c(L^1_z)}, \ \|f_0\|_{L^1_c(H^1_z)}\end{aligned}$$

$$\begin{aligned}\|\nabla S_2(t)Tf_0\|_{L^\infty} &= \|S_2(t)\nabla Tf_0\|_{L^\infty} = \|S_2(t)((f_0)_z, -c(f_0)_z)\|_{L^\infty} \\ &\leqslant \|S_1((1+c^2)t)((f_0)_z, -c(f_0)_z)\|_{L^1_c(L^\infty_z)} \\ &\lesssim \frac{1}{t^{1/2}}\|(f_0)_z\|_{L^1_c(L^1_z)}, \|f_0\|_{L^1_c(H^2_z)} + \|cf_0\|_{L^1_c(H^2_z)}.\end{aligned}$$

The Theorem will be proved if we show that the unique solution v of

$$iv_t + v_{xx} + v_{yy} + \lambda|v+\phi|^\sigma(v+\phi) = 0, \quad v(0) = v_0,\ \phi(t) = S_2(t)\phi_0$$

is global and remains small in the H^1 norm, for all $t > 0$. As before, we decompose the nonlinear term as

$$\lambda|v+\phi|^\sigma(v+\phi) = \sum_{i=0}^{5} g_i(v,\phi),$$

define h as in (8.1.6) and apply some Strichartz estimates in the Duhamel's formula for v in order to obtain inequality (8.1.8). The estimates for $i = 1, ..., 5$ are *mutatis mutandis* those that were derived for the numerable case, since one has control of the $W^{1,\infty}$ norm of $S_2\phi_0$. For the autonomous term, using the decay of $\|\phi\|_{W^{1,\infty}}$,

$$\||\phi|^5\|_{L^1(H^1)} \leqslant \int_0^t \|\phi(s)\|_{W^{1,\infty}}\|\phi(s)\|_{L^\infty}^2\|\phi(s)\|_4^2 \lesssim \|\phi\|_X^2\left(1 + \int_1^t \frac{1}{s^{3/2}}\right) \lesssim \epsilon^2$$

□

Theorem 8.4.8. *Set $\sigma \geqslant 3$. For any given $K > 0$ and $M > 1$, there exists $\phi_0 \in X \cap H^1(\mathbb{R}^2)$ satisfying the conditions of Theorem 8.4.7 and*

$$\|\phi_0\|_{L^2(\mathbb{R}^2)} > K, \quad \|\nabla\phi_0\|_{L^2(\mathbb{R}^2)} > K.$$

Consequently, for any $v_0 \in H^1(\mathbb{R}^2)$ such that

$$\|v_0\|_{H^1} < \epsilon(M)$$

the H^1 solution u of (NLS) with initial data $v_0 + \phi_0$ is global and satisfies

$$\|u - S_2\phi_0\|_{L^\infty((0,\infty),H^1(\mathbb{R}^2))} \leqslant \delta(\epsilon(M), M).$$

Proof of Theorem 8.4.8. Let $\psi \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in $[-1,1]$ and $\psi \equiv 0$ in $\mathbb{R}\backslash[-2,2]$. Take $g \in \mathcal{S}(\mathbb{R})$ such that

$$\|\mathcal{F}_\xi^{-1}\psi\|_{L^1}\|g\|_{W^{1,1}} + \|g\|_{H^2} < \sqrt{M},\ \|g\|_{H^1} < \sqrt{\epsilon(M)},\ |\mathcal{F}g(0)| > 0.$$

For any $\epsilon > 0$, define

$$\psi_\epsilon(\xi) = \psi\left(\frac{\xi}{\epsilon}\right), \quad g_\epsilon = g - \left(\mathcal{F}_\xi^{-1}\psi_\epsilon\right) \star_z g.$$

Then

$$\|g_\epsilon\|_{H^s} = \|(1+|\xi|^2)^{s/2}\mathcal{F}g_\epsilon\|_{L^2} \leqslant \|(1+|\xi|^2)^{s/2}\mathcal{F}g\|_{L^2} = \|g\|_{H^s},\ s = 1, 2.$$

and

$$\|g - g_\epsilon\|_{W^{1,1}} \leqslant \|\mathcal{F}_\xi^{-1}\psi_\epsilon\|_{L^1}\|g\|_{W^{1,1}} \leqslant \|\mathcal{F}_\xi^{-1}\psi\|_{L^1}\|g\|_{W^{1,1}} < \sqrt{M}.$$

Finally, take $f \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\|f\|_{L^1\cap L^\infty} < \sqrt{\epsilon},\ \|cf\|_{L^1} < \sqrt{M},\ \|cf\|_{L^2} > \frac{K}{\||\xi|^{1/2}\mathcal{F}g\|_{L^2(|\xi|>1)}}.$$

Defining $\phi_\epsilon = T(f \otimes g_\epsilon)$, it is now easy to check that ϕ_ϵ satisfies the conditions of Theorem 8.4.7. Moreover, by Corollary 8.2.10, for $\epsilon > 0$ small

$$\|\nabla\phi_\epsilon\|_{L^2} \geqslant \|cf\|_{L^2}\||\xi|^{1/2}\mathcal{F}g_\epsilon\|_{L^2} > \|cf\|_{L^2}\||\xi|^{1/2}\mathcal{F}g\|_{L^2(|\xi|>1)} > K.$$

All that is left is to prove that, for $\epsilon > 0$ small enough, $\|\phi_\epsilon\|_{L^2} > K$. This follows from

$$\|\phi_\epsilon\|_{L^2} = \|f\|_{L^2}\left\|\frac{\mathcal{F}g_\epsilon}{|\xi|^{1/2}}\right\|_2 \to \infty,\ \epsilon \to 0.$$

□

REMARK 8.4.2. The global H^1 solutions given by Theorem 8.4.8 have small $L^{\sigma+2}$ norm, since this norm is controlled by $\|v_0\|_{H^1} + \|\phi_0\|_X$. Therefore the energy of these solutions will always be positive and so there is no possible contradiction with the usual Virial blowup result by Glassey [35].

Lemma 8.4.9. *Suppose that* $f \in L^1_c(H^1_z) \cap L^\infty_c(L^2_z)$. *If*

$$zf, (1+c^2)f_z \in L^1_c(L^2_z) \cap L^\infty_c(L^2_z). \tag{8.4.5}$$

then, for some $C = C(f)$,

$$\|S_2(t)Tf\|_{L^{2p}} \lesssim \begin{cases} C & 2 \leqslant p \leqslant \infty \\ C(1+t^2) & 1 < p < 2 \end{cases}.$$

Proof. From Proposition 8.3.1 and Remark 8.2.1,

$$\begin{aligned}
&\|S_2(t)Tf\|_{L^4} = \|T(S_1((1+c^2)t)f(c))\|_{L^4} \\
\lesssim\ &\|S_1((1+c^2)t)f(c)\|_{L^1_c(L^2_z)} + \|S_1((1+c^2)t)f(c)\|_{L^\infty_c(L^2_z)} = \|f\|_{L^1_c(L^2_z)} + \|f\|_{L^\infty_c(L^2_z)}.
\end{aligned}$$

Furthermore, from Proposition 8.2.3,

$$\|S_2(t)Tf\|_{L^\infty} = \|T(S_1((1+c^2)t)f(c))\|_{L^\infty} \lesssim \|f\|_{L^1_c(H^1_z)}.$$

The estimate for $2 < p < \infty$ follows by interpolation. Now take $1 < p < 2$. By Proposition 8.2.3,

$$\|S_2(t)Tf\|_{L^{2p}} \leqslant \int \frac{1}{|c-c'|}\|S_1((1+c^2)t)f(c)\|_{L^p_z}\|S_1((1+(c')^2t))f(c')\|_{L^p_z}dcdc'.$$

Using Hölder's inequality,

$$\begin{aligned}
\|S_1((1+c^2)t)f(c)\|_{L^p_z} &\leqslant \left\|\frac{1}{1+|z|}\right\|_{L^{\frac{2p}{2-p}}} \|(1+|z|)S_1((1+c^2)t)f(c)\|_{L^2_z} \\
&\lesssim \|f(c)\|_{L^2_z} + \|zS_1((1+c^2)t)f(c)\|_{L^2_z}.
\end{aligned}$$

Recall the classical estimate, valid a.e. for $c \in \mathbb{R}$:

$$\|zS_1((1+c^2)t)f(c)\|_{L^2_z} \leqslant \|zf(c)\|_{L^2_z} + 2(1+c^2)t\|f_z(c)\|_{L^2_z}.$$

Hence, setting

$$\Theta(c,t) = \|f(c)\|_{L^2_z} + \|zf(c)\|_{L^2_z} + 2(1+c^2)t\|f_z(c)\|_{L^2_z},$$

we have

$$\|S_2(t)Tf\|_{L^{2p}} \lesssim \int \frac{1}{|c-c'|}\Theta(c,t)\Theta(c',t)dcdc'.$$

By a analogous reasoning to that of Remark 8.2.1 and (8.4.5),

$$\|S_2(t)Tf\|_{L^{2p}} \lesssim 1 + t^2.$$

□

Theorem 8.4.10. *Fix $\sigma = 1$. Consider $u_0 = v_0 + \phi_0$, $v_0 \in H^1(\mathbb{R}^2)$, $\phi_0 = Tf_0 \in X$. Assume that*

$$(1+c^3)(1+|z|)f_0 \in L^1_c(H^2_z) \cap L^\infty_c(H^2_z).$$

Then there exists a unique global solution $u \in C([0,\infty), E)$ of (NLS) with initial data u_0.

Proof of Theorem 8.4.10. The theorem will be proved once we show that the unique solution of

$$iv_t + v_{xx} + v_{yy} + \lambda|v+\phi|(v+\phi) = 0, \quad v(0) = v_0,\ \phi(t) = S_2(t)\phi_0 \tag{8.4.6}$$

is globally defined. In the following, we perform formal calculations which can be justified with suitable regularization arguments. Multiplying (8.4.6) by $\bar{v}$, integrating over $\mathbb{R}^2$ and taking the imaginary part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \lambda \operatorname{Im}\int |v+\phi|(v+\phi)\bar{v} = 0.$$

and so

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 \lesssim \|\phi\|_{L^\infty}\|v\|_{L^2}^2 + \|v\|_{L^2}\|\phi\|_{L^4}^2$$

Since, by the previous lemma, $\|\phi\|_{L^4}$ and $\|\phi\|_{L^\infty}$ are uniformly bounded, Gronwall's lemma implies that

$$\|v(t)\|_{L^2} \lesssim C(t),$$

where $C : \mathbb{R}^+ \to \mathbb{R}$ is a non decreasing continuous function. Now, multiplying (8.4.6) by $\overline{v}_t$, integrating and taking the real part,

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla v(t)\|_{L^2}^2 - \frac{\lambda}{3}\|v+\phi\|_{L^3}^3\right) = \operatorname{Im}\lambda\int \nabla\left(|v+\phi|(v+\phi)\right)\cdot\nabla\bar{\phi}.$$

Recalling that $\nabla\phi(t) = S_2T(f_z, -cf_z)$ and applying the previous lemma,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla v(t)\|_{L^2}^2 - \frac{\lambda}{3}\|(v+\phi)(t)\|_{L^3}^3\right)\\
&\lesssim \int |\nabla\phi(t)|^2|\phi(t)| + |\nabla\phi(t)||\phi(t)||v(t)| + |\nabla\phi(t)|^2|v(t)| + |\nabla\phi(t)||\nabla v(t)||v(t)|\\
&\lesssim \|\phi(t)\|_{L^4}\|\nabla\phi(t)\|_{L^{8/3}}^2 + \|\phi(t)\|_{L^4}\|\nabla\phi(t)\|_{L^4}\|v(t)\|_{L^2}\\
&\quad + \|\nabla\phi(t)\|_{L^4}^2\|v(t)\|_{L^2} + \|\nabla\phi(t)\|_{L^\infty}\|\nabla v(t)\|_{L^2}\|v(t)\|_{L^2}\\
&\lesssim (1+t)^4 + C(t) + C(t)\|\nabla v(t)\|_2.
\end{aligned}$$

Define the energy

$$E(t) = \frac{1}{2}\|\nabla v(t)\|_{L^2}^2 - \frac{\lambda}{3}\|v+\phi\|_{L^3}^3.$$

From Gagliardo-Nirenberg's inequality,

$$\|v+\phi\|_{L^3}^3 \lesssim \|v\|_{L^3}^3 + \|\phi\|_{L^3}^3 \lesssim \|\nabla v\|_{L^2}\|v\|_{L^2}^2 + (1+t)^6 \lesssim C(t)^2\|\nabla v\|_{L^2} + (1+t)^6.$$

Let T_{max} be the maximal time of existence of the solution of (8.4.6). Then, for any $0 < t_0 < t < T_{max}$,

$$\|\nabla v(t)\|_{L^2}^2 \lesssim |E(t_0)| + C(t)^2\|\nabla v(t)\|_{L^2} + (1+t)^6 + \int_{t_0}^t \left((1+s)^4 + C(s) + C(s)\|\nabla v(s)\|_2\right)ds.$$

Suppose, by contradiction, that $T_{max} < \infty$. Since $\|\nabla v(t)\|_{L^2} \to \infty$ as $t \to T_{max}$, one has, for $t > t_1$ sufficiently close to T_{max},

$$\|\nabla v(t)\|_{L^2}^2 \lesssim \int_{t_0}^t C(s)\|\nabla v(s)\|_2 ds.$$

However, this implies that $\|\nabla v(t)\|_{L^2}^2$ is bounded on $[t_1, T_{max}]$, which is absurd. □

Chapter 9

Finite speed of disturbance and other qualitative results

In this chapter, we discuss some properties of the nonlinear Schrödinger equation over $\mathbb{R}^d$

$$iu_t + \Delta u + \lambda |u|^\sigma u = 0, \quad \lambda \in \mathbb{R},\ 0 < \sigma < 4/(d-2)^+ \tag{NLS}$$

which can be used to obtain simple qualitative results. The first part will focus on the concept of finite speed of disturbance, while the second will deal with solutions of the (NLS) with infinite variance.

As it is well-known, the linear Schrödinger equation has *infinite speed of propagation*: the information at one point $x \in \mathbb{R}^d$ can influence points at arbitrary distance. This may be seen in the linear equation either by taking as initial condition the Dirac delta or by observing that the speed of a given frequency ξ is an unbounded function of ξ. In fact, it has been proven in [27] that the only solution of the linear Schrödinger equation with compact support at two different times is the zero solution.

Finite speed of propagation is a useful tool to obtain qualitative results on the dynamics of an equation, the classical example being the wave equation. One of these properties is *localization*, *i.e.*, to study what happens near a point $x \in \mathbb{R}^d$, one only needs to look at the backward light cone.

A similar application of finite speed of propagation is the *concatenation* of initial data: if one takes two compactly supported initial data u_0, v_0, then, given $v \in \mathbb{R}^d$, the solution with initial condition $u_0 + v_0(\cdot - v)$, $|v|$ large, will behave like the sum of the individual solutions up to a large time T_v. Moreover, if the nonlinear effects have already dissipated by time T_v, one expects that the solution is global. Results of this type are currently unavailable for the (NLS).

In this chapter, we give a weaker notion of speed of propagation and use it to obtain concatenation results for the (NLS). Finite speed of disturbance is an estimate for the amount of mass that appears in some observation set, taking into account the distance between that set and the initial support.

In the second part of the chapter, we look at solutions of the (NLS) with infinite variance. So far, blow-up results rely heavily on the Virial identity: defining the variance and the energy

$$V(w) = \|xw\|_2^2, \quad E(w) = \frac{1}{2}\|\nabla w\|_2^2 - \frac{1}{\sigma+2}\|w\|_{\sigma+2}^{\sigma+2},$$

one has $E(u(t)) = E(u_0)$ and $(V(u(t)))'' \leqslant 16E(u(t)) = 16E(u_0)$ for as long as u exists. Hence, if $V(u_0) < \infty$ and $E(u_0) < 0$, one sees that the solution u must cease to exist.

One of the natural questions that arises from this result is: do we really need $V(u_0) < \infty$? This seems natural, since the local well-posedness is posed over H^1 without any constraints on the variance. In fact, in [60], it was proved that the finite variance condition could be replaced by the radiality of the initial data u_0. The only result (to our knowledge) that overcomes somehow this problem has appeared on [42], for the cubic (NLS) in dimension 3: for any initial data u_0 with negative energy, the corresponding solution cannot be H^1-bounded uniformly in time. Here, we shall consider the general L^2-(super)critical case and build examples of nonradial unbounded H^1 solutions.

9.1 Finite speed of disturbance

Let us start with the linear equation

$$iu_t + \Delta u = 0, \quad u(0) = u_0 \in H^1(\mathbb{R}^d). \tag{LS}$$

If one takes $\phi \in W^{1.\infty}(\mathbb{R}^d)$ real-valued, then

$$\frac{1}{2}\frac{d}{dt}\int \phi^2|u|^2 = 2\,\mathrm{Im}\int \phi\bar{u}\nabla\phi\cdot\nabla u \leqslant 2\|\phi u\|_2\|\nabla\phi\|_\infty\|\nabla u\|_2.$$

Integrating this differential inequality,

$$\|\phi u(t)\|_2 \leqslant \|\phi u_0\|_2 + 2t\|\nabla\phi\|_\infty \sup_{s\in[0,t]}\|\nabla u(s)\|_2 = \|\phi u_0\|_2 + 2t\|\nabla\phi\|_\infty\|\nabla u_0\|_2,$$

since the L^2 norm of the gradient is preserved by (LS). Now take two disjoint smooth open sets $A, B \subset \mathbb{R}^d$ and take $\phi \in W^{1,\infty}$ such that

$$\phi \equiv 1 \text{ on } B, \quad \phi \equiv 0 \text{ on } A, \quad \|\nabla\phi\|_\infty < \frac{1}{\mathrm{dist}(A,B)}.$$

Then

$$\|u(t)\|_{L^2(B)} \leqslant \frac{2t\|\nabla u_0\|_2}{\mathrm{dist}(A,B)} + \|u_0\|_{L^2(\mathbb{R}^d\setminus A)}, \quad A, B \subset \mathbb{R}^d.$$

In the special case where $\phi u_0 \equiv 0$, one has

$$\|\phi u(t)\|_2 \leqslant 2\|\nabla\phi\|_\infty\|\nabla u_0\|_2 t.$$

and so one obtains the *finite speed of disturbance* for the (LS):

$$\|u(t)\|_{L^2(B)} \leqslant \frac{2\|\nabla u_0\|_2}{\mathrm{dist}(A,B)}t. \tag{9.1.1}$$

This inequality tells us that, even though information may travel at any speed, the *amount of information* that reaches some set B grows (at most) linearly in time, with growth factor inversely proportional to the distance between the source of the information and the observation set. In another way, even though the higher frequencies travel faster, they carry a controlled amount of mass.

REMARK 9.1.1. Fix $t \geqslant 0$. Given any $\gamma(t) \geqslant 1$, the choice $B = \mathbb{R}^d \backslash (A + B_{\gamma(t)}(0))$ in (9.1.1) yields

$$\|u(t)\|_{L^2(\mathbb{R}^d \backslash (A+B_{\gamma(t)}(0)))} \leqslant \frac{2\|\nabla u_0\|_2 t}{\gamma(t)}.$$

For any $\epsilon > 0$, if $\gamma(t) = \gamma t$, $\gamma = 2\|\nabla u_0\|_2/\epsilon$, we see that

$$\|u(t)\|_{L^2(\mathbb{R}^d \backslash (A+B_{\gamma t}(0)))} \leqslant \epsilon.$$

This means that most of the total mass lies inside a specific cone of light, with speed given by the initial kinetic energy.

REMARK 9.1.2. Suppose that B is such that, for some unit vector $v \in \mathbb{R}^d$,

$$\mathrm{dist}(A, B + vt) = \mathrm{dist}(A, B) + t, \quad t > 0.$$

Using the Galilean invariance

$$u_b(x,t) = e^{i\frac{bv}{2}\left(x - \frac{bv}{2}t\right)} u(t, x - bvt), \quad b > 0,$$

one has, for any set C with $\mathrm{dist}(A, C) > 0$,

$$\|u_b(t)\|_{L^2(C)} \leqslant \frac{2\|\nabla u_b(0)\|_2}{\mathrm{dist}(A,C)} t.$$

For each fixed $t > 0$, take $C = B + bvt$. One then arrives at a more general estimate for the speed of disturbance

$$\|u(t)\|_{L^2(B)} = \|u_v(t)\|_{L^2(B+bvt)} \leqslant \frac{t}{\mathrm{dist}(A,B) + bt} \left(4\|\nabla u_0\|_2^2 + b^2\|u_0\|_2^2\right)^{1/2}, \quad b > 0.$$

Notice that, taking both $b, t \to \infty$, one has the trivial bound $\|u(t)\|_{L^2(B)} \leqslant \|u_0\|_2$.

In the (NLS) case, analogous computations yield

$$\|\phi u(t)\|_2 \leqslant 2\|\nabla\phi\|_\infty \left(\sup_{s\in[0,t]} \|\nabla u(s)\|_2 \right) t. \tag{9.1.2}$$

Choosing ϕ as in the linear case, one has

$$\|u(t)\|_{L^2(B)} \leqslant \frac{2t \sup_{s\in[0,t]} \|\nabla u(s)\|_2}{\mathrm{dist}(A,B)} + \|u_0\|_{L^2(\mathbb{R}^d\backslash A)}, \quad A, B \subset \mathbb{R}^d.$$

in the general case and, if $\operatorname{supp} u_0 \subset A$, one obtains the finite speed of disturbance for the (NLS)

$$\|u(t)\|_{L^2(B)} \leqslant \frac{2\sup_{s\in[0,t]}\|\nabla u(s)\|_2}{\operatorname{dist}(A,B)}t.$$

In this case, we see that, as long as u remains bounded in H^1, the solution will have finite speed of disturbance and the previous considerations are still valid. Another useful estimate can be obtained from (9.1.2): using Gagliardo-Nirenberg's inequality,

$$\begin{aligned}
\|u(t)\|_{L^{\sigma+2}(B)} &\leqslant \|\phi u(t)\|_{\sigma+2} \lesssim \|\phi u(t)\|_2^{1-\frac{d\sigma}{2(\sigma+2)}}\|\nabla(\phi u(t))\|_2^{\frac{d\sigma}{2(\sigma+2)}} \\
&\lesssim \left(\frac{2t}{\operatorname{dist}(A,B)}\sup_{s\in[0,t]}\|\nabla u(s)\|_2\right)^{1-\frac{d\sigma}{2(\sigma+2)}}\|\nabla(\phi u(t))\|_2^{\frac{d\sigma}{2(\sigma+2)}} \qquad (9.1.3)\\
&\lesssim \left(\frac{2t}{\operatorname{dist}(A,B)}\sup_{s\in[0,t]}\|\nabla u(s)\|_2\right)^{1-\frac{d\sigma}{2(\sigma+2)}} \\
&\quad\times\left(\|\nabla u(t)\|_2^2+\frac{1}{\operatorname{dist}(A,B)^2}\|u_0\|_2^2\right)^{\frac{d\sigma}{4(\sigma+2)}}.
\end{aligned}$$

Remark 9.1.3. Consider the defocusing case $\lambda < 0$ in the L^2-critical case. As it is well-known, given any initial data $u_0 \in H^1(\mathbb{R}^d) \cap L^2(|x|^2dx) =: \Sigma$, the corresponding solution of (NLS) u scatters to a linear solution, *i.e.*, there exists a unique $u_+ \in \Sigma$ such that

$$\|S(-t)u(t)-u_+\|_\Sigma \to 0, \quad t \to \infty.$$

Thus one may define the forward scattering operator as the mapping $u_0 \mapsto u_+$. We observe that the simple application of finite speed of disturbance can lead to an estimate for the scattering operator. In fact, using the *lens transform* (see [11] and Appendix B), if v is the solution of the nonlinear Schrödinger equation with an harmonic potential

$$iv_t + \Delta v - |x|^2 v + \lambda|v|^{4/d}v = 0, \quad v(0) = u_0,$$

then the Fourier transform $\hat{u}_+$ of u_+ is precisely $v(\pi/2)$ (this has been observed in [70]). This equation also enjoys finite speed of disturbance:

$$\|v(t)\|_{L^2(B)} \leqslant \frac{2t\sup_{s\in[0,t]}\|\nabla v(s)\|_2}{\operatorname{dist}(A,B)} + \|u_0\|_{L^2(\mathbb{R}^d\setminus A)}, \quad A,B\subset\mathbb{R}^d.$$

Moreover, by conservation of energy, one has

$$\|\nabla v(t)\|_2 \lesssim \|v_0\|_\Sigma = \|u_0\|_\Sigma,\ t>0.$$

Thus, taking $t=\pi/2$, one obtains

$$\|\hat{u}_+\|_{L^2(B)} \leqslant \frac{\pi\|u_0\|_\Sigma}{\operatorname{dist}(A,B)} + \|u_0\|_{L^2(\mathbb{R}^d\setminus A)}, \quad A,B\subset\mathbb{R}^d.$$

In particular, the localization of the initial data on the physical side implies a localization of the scattering state u_+ on the frequency side.

Remark 9.1.4. Finite speed of perturbation, being a weaker version of the classical finite speed of propagation, can be observed in a larger number of equations. In particular, it would be interesting to study this concept for other dispersive PDE's for which one has infinite speed of propagation, such as the Korteweg-de-Vries equation.

With the concept of finite speed of disturbance at hand, we state the main result of this section. Since global existence in the L^2-subcritical case is already known for any initial data, we focus on the L^2-(super)critical case $\sigma \geqslant 4/d$. In what follows, $\mathrm{NLS}(u_0)$ will denote the maximal solution of (NLS) with initial condition u_0. We recall the Besov space $B^s_{\rho,2}(\mathbb{R}^d)$, which is the closure of $\mathcal{D}(\mathbb{R}^N)$ with the norm

$$\|u\|^2_{B^s_{\rho,2}} = \|u\|^2_\rho + \|u\|^2_{\dot{B}^s_{\rho,2}} := \|u\|^2_\rho + \int_0^\infty \left(\tau^{-s} \sup_{|y|<\tau} \|u(\cdot - y) - u\|_\rho \right)^2 \frac{d\tau}{\tau}.$$

Given any time interval I, we set

$$\|u\|_{S^s(I)} = \|u\|_{L^\infty(I,H^s)} + \|u\|_{L^\gamma(I,B^s_{\rho,2})}, \quad \rho = \frac{d(\sigma+2)}{d+s\sigma}, \ \gamma = \frac{4(\sigma+2)}{\sigma(d-2s)}, \ 0 < s < 1,$$

and, for $q = 4(\sigma+2)/\sigma d$,

$$\|u\|_{S^0(I)} = \|u\|_{L^\infty(I,L^2)} + \|u\|_{L^{\sigma+2}(I,L^{\sigma+2})}, \quad \|u\|_{S^1(I)} = \|u\|_{L^\infty(I,H^1)} + \|u\|_{L^q(I,W^{1,\sigma+2})}$$

Define the set of global decaying solutions with bounded Strichartz norms (up to order one) as

$$\mathcal{GD} = \left\{ u_0 \in H^1(\mathbb{R}^d) : T(\mathrm{NLS}(u_0)) = \infty, \ \|\mathrm{NLS}(u_0)\|_{S^1(0,\infty)} < \infty \right\}.$$

As proven in [12, Theorem 6.2.1], there exists $\delta > 0$ such that

$$\{u_0 \in H^1(\mathbb{R}^d) : \|u_0\|_{H^1} < \delta\} \subset \mathcal{GD}.$$

Theorem 9.1.1 (Concatenation of initial data)**.** *Set $\sigma = 4/d$ or $\sigma \geqslant \min\{1, 4/d\}$. Given initial data $u_0, v_0 \in H^1$, a fixed time $T < T(u_0), T(v_0)$ and $\epsilon > 0$, there exists $D_T > 0$ such that, for any $w_0 \in H^1$ small enough,*

$$T(u_0 + v_0(\cdot - y) + w_0) > T, \quad |y| > D_T$$

and, taking s such that $\sigma = 4/(d-2s)$,

$$\|\mathrm{NLS}(u_0 + v_0(\cdot - y) + w_0) - \mathrm{NLS}(u_0) - \mathrm{NLS}(v_0(\cdot - y))\|_{S^s(0,T)} < \epsilon.$$

Moreover, if $u_0, v_0 \in \mathcal{GD}$, there exists $D_\infty > 0$ such that $u_0 + v_0(\cdot - y) + w_0 \in \mathcal{GD}$, $|y| > D_\infty$, and

$$\|\mathrm{NLS}(u_0 + v_0(\cdot - y) + w_0) - \mathrm{NLS}(u_0) - \mathrm{NLS}(v_0(\cdot - y))\|_{S^s(0,\infty)} < \epsilon.$$

Before we prove Theorem 9.1.1 (which will be done first for the critical case and then for the supercritical one), we make a few comments.

Remark 9.1.5. The value of D_∞ depends only on the global bound M for NLS(u_0) and NLS(v_0) and the size of the supports of the initial data. One may extend the above result for general (not compactly supported) initial data. In that case, the value of d_∞ will depend of the specific shape of the initial data.

Remark 9.1.6. Independently of the spatial dimension, the result is always true for $\sigma = 4/d$. In the supercritical case, one could try to prove the concatenation result with similar arguments to those of [12, Theorem 6.2.1]. However, the information given by the finite speed of disturbance concerns the solution itself and not its derivatives. As a consequence, one must try to prove global well-posedness by using as little derivatives as possible. This is achieved over the critical space H^s, with $\sigma = 4/(d-2s)$ and $s > 0$. To estimate properly the interaction between the two solutions in Besov spaces, we require that $\sigma \geqslant 1$.

Remark 9.1.7. Observe that the concatenation of two compactly supported initial data with positive energy also has positive energy. Thus the concatenation result does not contradict the Virial blow-up argument.

It is important to notice that the second part of Theorem 9.1.1 can be iterated: one starts with two global solutions with linear decay and builds a new global solution with linear decay. Moreover, the L^2 and H^1 norms of the new initial data is the sum of the corresponding norms of the given data. As a consequence, one obtains

Corollary 9.1.2. *Set $\sigma = 4/d$ or $1, 4/d \leqslant \sigma$. Given $K_1, K_2 > 0$, there exists $u_0 \in \mathcal{GD}$ compactly supported such that*

$$\|u_0\|_2 = K_1, \quad \|\nabla u_0\|_2 = K_2.$$

Our result indicates that blow-up behaviour is necessarily connected with how much localized is the initial data: if the initial data is made up of small H^1 pieces, sufficiently spread out in space, the corresponding solution is global. On the other hand, we point out that the Virial blow-up argument is also connected with the localization of the initial data, through both variance and energy. Though far from concrete, an underlying necessary and sufficient condition for blow-up becomes apparent.

Remark 9.1.8. The question of concatenation of global solutions with no decay properties is not a trivial matter. The argument used to prove the concatenation result still works if we add small H^1 perturbations to the initial data. Consequently, such a procedure cannot be applied to generic global solutions: if so, one would prove that, around the concatenation of the ground-state with the zero solution (which is again the ground-state), any initial data gives rise to global solutions, which is false (see [6]).

Proof of Theorem 9.1.1 for $\sigma = 4/d$. Step 1. Before we proceed, we make a number of simplifications using the symmetries of the (NLS). First, we may consider $y = 2De_1$, where $e_1 \in \mathbb{R}^d$ is the first element of the canonic basis of $\mathbb{R}^d$. Moreover, due to the translation invariance, we may prove the result for the initial data $u_0(\cdot + y/2) + v_0(\cdot - y/2) = u_0(\cdot + De_1) + v_0(\cdot - De_1)$. In what follows, $\delta(D)$ will denote a decreasing function such that

$$\delta(D) \to 0, \quad D \to \infty.$$

Define $A^- = \{x \in \mathbb{R}^d : x_1 < -D/2\}$ and $A^+ = \{x \in \mathbb{R}^d : x_1 > D/2\}$. Since $u_0, v_0 \in L^2$,

$$\|u_0(\cdot + De_1)\|_{L^2(\mathbb{R}^d \setminus A^-)}, \ \|v_0(\cdot - De_1)\|_{L^2(\mathbb{R}^d \setminus A^+)} < \delta(D). \tag{9.1.4}$$

Set $u = \mathrm{NLS}(u_0(\cdot + De_1))$, $v = \mathrm{NLS}(v_0(\cdot - De_1))$ and consider the initial value problem

$$iw_t + \Delta w + |u + v + w|^\sigma (u + v + w) - |u|^\sigma u - |v|^\sigma v = 0, w(0) = w_0 \in H^1.$$

Notice that

$$\mathrm{NLS}(u_0(\cdot + De_1) + v_0(\cdot - De_1) + w_0) = u + v + w$$

for as long as any three solutions exist. As a consequence, the local existence of w as a L^2-solution is a trivial matter.

Step 2. Since $T < T(u_0), T(v_0)$, there exists $M > 0$ such that

$$\|u\|_{L^\infty((0,T),H^1)}, \|v\|_{L^\infty((0,T),H^1)} \leqslant M. \tag{9.1.5}$$

One has

$$||u + v + w|^\sigma (u + v + w) - |u|^\sigma u - |v|^\sigma v| \lesssim |v|^\sigma |u| + |u|^\sigma |v| + |u|^\sigma |w| + |v|^\sigma |w| + |w|^{\sigma+1}$$

We write $B^+ = \{x \in \mathbb{R}^d : x_1 > 0\}$ and $B^- = \{x \in \mathbb{R}^d : x_1 < 0\}$. The idea is that, due to (9.1.4), u is small over B^+ and v is small over B^-.

Setting $\rho = (\sigma + 2)/(\sigma + 1)$,

$$\begin{aligned}
\||u|^\sigma v\|_{L^\rho} &\lesssim \||u|^\sigma v\|_{L^\rho(B^+)} + \||u|^\sigma v\|_{L^\rho(B^-)} \\
&\lesssim \|u\|^\sigma_{L^{\sigma+2}(B^+)} \|v\|_{L^{\sigma+2}(B^+)} + \|u\|^\sigma_{L^{\sigma+2}(B^-)} \|v\|_{L^{\sigma+2}(B^-)} = I_1 + I_2
\end{aligned}$$

We estimate I_1: we use (9.1.3) on the first term and the uniform bound (9.1.5),

$$\begin{aligned}
I_1 \lesssim \|u\|^\sigma_{L^{\sigma+2}(B^+)} \|v\|_{L^{\sigma+2}} &\lesssim \left(\frac{TM}{\mathrm{dist}(A^-, B^+)} + \|u_0\|_{L^2(\mathbb{R}^d \setminus A^-)} \right)^{\frac{\sigma^2}{\sigma+2}} M^{\frac{2\sigma}{\sigma+2}+1} \\
&\lesssim \left(\frac{TM}{D} + \delta(D) \right)^{\frac{\sigma^2}{\sigma+2}} M^{\frac{2\sigma}{\sigma+2}+1} \\
&\lesssim \delta(D).
\end{aligned}$$

The same reasoning can be applied to I_2, which implies that $\||u|^\sigma v\|_{L^\rho} < \delta(D)$. Analogously, we also have $\||v|^\sigma u\|_{L^\rho} \lesssim \delta(D)$.

We apply Strichartz estimates to the Duhamel formula for w on a fixed interval $(0,t)$:

$$\begin{aligned}
\|w\|_{S^0(0,t)} &\lesssim \|w_0\|_2 + t^{\frac{1}{\rho}}\delta(D) + \||u|^\sigma|w|\|_{L^\rho((0,t),L^\rho)} \\
&+ \||v|^\sigma|w|\|_{L^\rho((0,t),L^\rho)} + \|w\|^{\sigma+1}_{S^0(0,t)} \\
&\lesssim \|w_0\|_2 + t^{\frac{1}{\rho}}\delta(D) + \|u\|^\sigma_{L^{\sigma+2}((0,t),L^{\sigma+2})}\|w\|_{S^0(0,t)} \\
&+ \|v\|^\sigma_{L^{\sigma+2}((0,t),L^{\sigma+2})}\|w\|_{S^0(0,t)} + \|w\|^{\sigma+1}_{S^0(0,t)} \\
&\lesssim \|w_0\|_2 + t^{\frac{1}{\rho}}\delta(D) + t^{\frac{\sigma}{\sigma+2}}M^\sigma\|w\|_{S^0(0,T)} + \|w\|^{\sigma+1}_{S^0(0,t)}.
\end{aligned}$$

We choose T_0 such that

$$T_0^{\frac{\sigma}{\sigma+2}}M^\sigma \lesssim \frac{1}{2},$$

so that

$$\|w\|_{S^0(0,t)} \lesssim \|w_0\|_2 + \delta(D) + \|w\|^{\sigma+1}_{S^0(0,t)}, \quad t < T_0.$$

A standard obstruction argument then implies that there exists $\eta > 0$ small such that, if

$$\|w_0\|_2 + \delta(D) < \eta',\ \eta' < \eta,$$

then w exists (as an L^2 solution) up to time T_0 and $\|w(T_0)\|_2 < 2\eta'$. This process may be iterated as long as the L^2 norm of w remains below η. Thus, for sufficiently small $\|w_0\|_2$ and large D, one guarantees that w exists up to time T and that

$$\|w\|_{S^0(0,T)} < \eta < \epsilon.$$

This concludes the proof of the first part of Theorem 9.1.1.

Step 3. For the second part of the Theorem, fix $\delta > 0$ small and choose T large enough such that

$$\|u\|_{S^0(T,\infty)}, \|v\|_{S^0(T,\infty)} \leqslant \delta.$$

Applying the first part of the Theorem, for D_T large, w is defined up to time T with

$$\|w\|_{S^0(0,T)} \leqslant \delta.$$

Recalling that $\rho = (\sigma+2)/(\sigma+1)$, one easily checks that

$$\||u|^\sigma v\|_{L^\rho((T,\infty),L^\rho)} \lesssim \|u\|^\sigma_{S^0(T,\infty)}\|v\|_{S^0(T,\infty)} \leqslant \delta,$$

$$\||v|^\sigma u\|_{L^\rho((T,\infty),L^\rho)} \lesssim \|v\|^\sigma_{S^0(T,\infty)}\|u\|_{S^0(T,\infty)} \leqslant \delta.$$

Let T^* be the maximal time of existence of w. Applying Strichartz estimates to the Duhamel formula

$$w(t) = S(t-T)w(T) - i\lambda\int_T^t S(t-s)\left(|u+v+w|^\sigma(u+v+w) - |u|^\sigma u - |v|^\sigma v\right)ds,$$

with $T < t < T^*$, one has

$$\begin{aligned}
\|w\|_{S^0(T,t)} &\lesssim \|w(T)\|_2 + \delta + \||u|^\sigma |w|\|_{L^{\rho}((T,t),L^\rho)} \\
&+ \||v|^\sigma |w|\|_{L^{\rho}((T,t),L^\rho)} + \|w\|^{\sigma+1}_{S^0(T,t)} \\
&\lesssim \delta + \|u\|^\sigma_{L^{\sigma+2}((T,t),L^{\sigma+2})}\|w\|_{S^0(T,t)} \\
&+ \|v\|^\sigma_{L^{\sigma+2}((T,t),L^{\sigma+2})}\|w\|_{S^0(T,t)} + \|w\|^{\sigma+1}_{S^0(T,t)} \\
&\lesssim \delta + \delta\|w\|_{S^0(T,t)} + \|w\|^{\sigma+1}_{S^0(T,t)}, \quad T < t < T^*.
\end{aligned}$$

Thus

$$\|w\|_{S^0(T,t)} \lesssim \delta + \|w\|^{\sigma+1}_{S^0(T,t)}, \quad T < t < T^*,$$

which, for δ sufficiently small, implies that

$$\|w\|_{S^0(T,t)} < \epsilon,\ T < t < T^*.$$

The blow-up alternative (on L^2) now implies that $T^* = \infty$ and that $\|w\|_{S^0(0,\infty)} < \epsilon$. Moreover, since the initial data is in H^1, by persistence of regularity, w is globally defined in H^1. $\square$

The remainder of this section will focus on the proof of Theorem 9.1.1 for the supercritical case. To that end, we shall work on H^s, with $\sigma = 4/(d-2s)$. An admissible pair of particular importance is

$$\rho = d(\sigma+2)/(d+s\sigma), \quad \gamma = \frac{4(\sigma+2)}{\sigma(d-2s)} = \sigma + 2.$$

Lemma 9.1.3. *One has*

$$W^{1,\rho}(\mathbb{R}^d) \hookrightarrow B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^{\frac{\sigma\rho\rho'}{\rho-\rho'}}(\mathbb{R}^d).$$

Proof. The second injection is a direct consequence of Sobolev's injection. For the first, we take $u \in C_0^\infty(\mathbb{R}^d)$ and write

$$\|u\|^2_{B^s_{\rho,2}} \lesssim \|u\|^2_\rho + \int_0^1 \left(\tau^{-s}\sup_{|y|<\tau}\|u(\cdot - y) - u\|_\rho\right)^2 \frac{d\tau}{\tau} + \int_1^\infty \left(\tau^{-s}\sup_{|y|<\tau}\|u(\cdot - y) - u\|_\rho\right)^2 \frac{d\tau}{\tau}$$

The characterization of Sobolev spaces using translation operators (see, for example, [8, Proposition 8.5]) implies that

$$\|u(\cdot - y) - u\|_\rho \lesssim \|\nabla u\|_\rho |y|$$

Hence

$$\|u\|^2_{B^s_{\rho,2}} \lesssim \|u\|^2_\rho + \int_0^1 \left(\tau^{-s}\sup_{|y|<\tau}\|\nabla u\|_\rho |y|\right)^2 \frac{d\tau}{\tau} + \int_1^\infty \left(\tau^{-s}\sup_{|y|<t}\|u\|_\rho\right)^2 \frac{d\tau}{\tau}$$

$$\lesssim \|u\|_{W^{1,\rho}}^2 \left(1 + \int_0^1 \frac{d\tau}{\tau^{1-2s}} + \int_1^\infty \frac{d\tau}{\tau^{1+s}}\right) \lesssim \|u\|_{W^{1,\rho}}^2.$$

□

As in the first step of the proof of Theorem 9.1.1 in the critical case, from now on, $\delta(D)$ will denoted a decreasing function of D such that $\delta(D) \to 0$ as $D \to \infty$.

Lemma 9.1.4. *Fix $\sigma > 4/d$ such that $\sigma \geqslant 1$. Given $u_0, v_0 \in H^1$ and $T < T(u_0), T(v_0)$, if one writes $u = \mathrm{NLS}(u_0(\cdot + De_1))$ and $v = \mathrm{NLS}(v_0(\cdot - De_1))$, then*

$$\||u+v|^\sigma(u+v) - |u|^\sigma u - |v|^\sigma v\|_{L^{\gamma'}((0,T),B^s_{\rho',2})} \leqslant \delta(D).$$

Proof. Set

$$N(u,v) = |u+v|^\sigma(u+v) - |u|^\sigma u - |v|^\sigma v \tag{9.1.6}$$

and

$$M = \|u\|_{L^\infty((0,T),H^1)} + \|u\|_{L^\gamma((0,T),W^{1,\rho})} + \|v\|_{L^\infty((0,T),H^1)} + \|v\|_{L^\gamma((0,T),W^{1,\rho})},$$

$$B(t) = \|u(t)\|_{W^{1,\rho}} + \|v(t)\|_{W^{1,\rho}}.$$

It is easy to check, using Hölder's inequality, that

$$\|B(t)^{\sigma+1}\|_{L^{\gamma'}(0,T)} \lesssim T^{\frac{4-\sigma(d-2s)}{4}} M^{\sigma+1}$$

Once again, write

$$A^- = \{x \in \mathbb{R}^d : x_1 < -D/2\}, \quad A^+ = \{x \in \mathbb{R}^d : x_1 > D/2\},$$

$$B^- = \{x \in \mathbb{R}^d : x_1 < 0\}, \quad B^+ = \{x \in \mathbb{R}^d : x_1 > 0\}.$$

Since $u_0, v_0 \in L^2$,

$$\|u_0(\cdot + De_1)\|_{L^2(\mathbb{R}^d \setminus A^-)},\ \|v_0(\cdot - De_1)\|_{L^2(\mathbb{R}^d \setminus A^+)} < \delta(D).$$

We set $0 < a, b < 1$ such that

$$\|z\|_{L^\alpha} \lesssim \|z\|_{L^\rho}^a \|\nabla z\|_{L^\rho}^{1-a}, \quad \|z\|_{L^\rho} \lesssim \|z\|_{L^2}^b \|\nabla z\|_{L^2}^{1-b}, \quad \alpha = \sigma\rho\rho'/(\rho - \rho'),\ z \in C_0^\infty(\mathbb{R}^d).$$

Then

$$\begin{aligned}\|N(u,v)\|_{\dot{B}^s_{\rho',2}}^2 &= \int_0^\infty \left(\tau^{-s} \sup_{|y|<\tau} \|N(u,v)(\cdot - y) - N(u,v)\|_{\rho'}\right)^2 \frac{d\tau}{\tau} \\ &\lesssim \int_0^1 \left(\tau^{-s} \sup_{|y|<\tau} \|N(u,v)(\cdot - y) - N(u,v)\|_{L^{\rho'}(B^+)}\right)^2 \frac{d\tau}{\tau}\end{aligned}$$

$$+\int_0^1\left(\tau^{-s}\sup_{|y|<\tau}\|N(u,v)(\cdot-y)-N(u,v)\|_{L^{\rho'}(B^-)}\right)^2\frac{d\tau}{\tau}$$
$$+\|N(u,v)\|^2_{L^{\rho'}(B^+)}+\|N(u,v)\|^2_{L^{\rho'}(B^-)}\lesssim I_1^2+I_2^2+I_3^2+I_4^2.$$

We treat I_1 as follows: setting $C^\pm=B^\pm+B_1(0)$ and recalling that $W^{1,\rho}\hookrightarrow B^s_{\rho,2}\hookrightarrow L^\alpha$, $\alpha=\sigma\rho\rho'/(\rho-\rho')$,

$$\begin{aligned}\|N(u,v)(\cdot-y)-N(u,v)\|_{L^{\rho'}(B^+)}&\lesssim\left(\|u\|^{\sigma-1}_{L^\alpha(C^+)}+\|v\|^{\sigma-1}_{L^\alpha(C^+)}\right)\\&\times\left(\|u\|_{L^\alpha(C^+)}\|v(\cdot-y)-v\|_{L^\rho(B^+)}+\|v\|_{L^\alpha(C^+)}\|u(\cdot-y)-u\|_{L^\rho(B^+)}\right)\\&\lesssim B(t)^{\sigma-1}\|u\|_{L^\alpha(C^+)}\|v(\cdot-y)-v\|_{L^\rho}+B(t)^\sigma\|u(\cdot-y)-u\|_{L^\rho(B^+)}\end{aligned}$$

For the first term, take a smooth cut-off function ϕ with $\phi\equiv 1$ over C^+ and $\phi\equiv 0$ over A^-. Then, from Gagliardo-Nirenberg and finite speed of disturbance,

$$\begin{aligned}\|u\|_{L^\alpha(C^+)}&\lesssim\|\phi u\|_{L^\alpha}\lesssim\|\phi u\|^a_{L^\rho}\|\nabla(\phi u)\|^{1-a}_{L^\rho}\lesssim\|\phi u\|^{ab}_{L^2}\|\nabla(\phi u)\|^{a(1-b)}_{L^2}\|u\|^{1-a}_{W^{1,\rho}}\\&\lesssim\|\phi u\|^{ab}_{L^2}M^{a(1-b)}B(t)^{1-a}\lesssim\left(\frac{TM}{D}+\|u_0\|_{L^2(\mathbb{R}^d\setminus A^-)}\right)^{ab}M^{a(1-b)}B(t)^{1-a}\\&\lesssim\delta(D)B(t)^{1-a}\end{aligned}$$

For the second term, taking $\epsilon>0$ small, for $|y|<1$,

$$\begin{aligned}\|u(\cdot-y)-u\|_{L^\rho(B^+)}&=\|u(\cdot-y)-u\|^\epsilon_{L^\rho(B^+)}\|u(\cdot-y)-u\|^{1-\epsilon}_{L^\rho(B^+)}\\&\lesssim\left(\|u\|_{L^\rho(C^+)}\right)^\epsilon\left(\|u(\cdot-y)-u\|_{L^\rho}\right)^{1-\epsilon}\\&\lesssim\left(\|\phi u\|^b_{L^2}\|\nabla(\phi u)\|^{1-b}_{L^2}\right)^\epsilon\left(\|\nabla u\|_{L^\rho}|y|\right)^{1-\epsilon}\\&\lesssim\delta(D)B(t)^{1-\epsilon}|y|^{1-\epsilon}\end{aligned}$$

Hence

$$\begin{aligned}|I_1|&\lesssim\delta(D)B(t)^{\sigma-a}\|v\|_{B^s_{\rho,2}}+\delta(D)B(t)^{\sigma+1-\epsilon}\left(\int_0^1\frac{1}{\tau^{1+2s-2(1-\epsilon)}}d\tau\right)^{1/2}\\&\lesssim\delta(D)\left(B(t)^{\sigma+1-a}+B(t)^{\sigma+1-\epsilon}\right)\\&\lesssim\delta(D)(1+B(t)^{\sigma+1})\end{aligned}$$

and so
$$\|I_1\|_{L^{\gamma'}(0,T)}\lesssim\delta(D)(T^{\frac{1}{\gamma'}}+T^{\frac{4-(d-2s)\sigma}{4}})\lesssim\delta(D).$$

For the I_3 term,

$$\begin{aligned}|I_3|&\lesssim\||u|^\sigma v\|_{L^{\rho'}(B^+)}+\||v|^\sigma u\|_{L^{\rho'}(B^+)}\lesssim\|u\|^\sigma_{L^\alpha(B^+)}\|v\|_\rho+\|v\|^\sigma_\alpha\|u\|_{L^\rho(B^+)}\\&\lesssim\left(\delta(D)B(t)^{1-a}\right)^\sigma\|v\|_{W^{1,\rho}}+\|v\|^\sigma_{W^{1,\rho}}\delta(D)\\&\lesssim\delta(D)B(t)^{\sigma(1-a)+1}+\delta(D)B(t)^\sigma\lesssim\delta(D)(1+B(t)^{\sigma+1}).\end{aligned}$$

As for the I_1 term, this implies $\|I_3\|_{L^{\gamma'}(0,T)} \lesssim \delta(D)$. The estimates for I_2 and I_4 are analogous. Thus

$$\|N(u,v)\|_{L^{\gamma'}((0,T),\dot{B}^s_{\rho',2})} \lesssim \delta(D).$$

Finally,

$$\begin{aligned}\|N(u,v)\|_{L^{\gamma'}((0,T),B^s_{\rho',2})} &\lesssim \|N(u,v)\|_{L^{\gamma'}((0,T),L^{\rho'})} + \|N(u,v)\|_{L^{\gamma'}((0,T),\dot{B}^s_{\rho',2})}\\ &\lesssim \|I_3\|_{L^{\gamma'}(0,T)} + \|I_4\|_{L^{\gamma'}(0,T)} + \delta(D) \lesssim \delta(D).\end{aligned}$$

□

Proof of Theorem 9.1.1 for $\sigma > 4/d$. We follow closely the proof of the L^2-critical case $\sigma = 4/d$. Set $y = 2De_1$, $u = \mathrm{NLS}(u_0(\cdot + y/2))$ and $v = \mathrm{NLS}(v_0(\cdot - y/2))$. Once again, consider the initial value problem

$$iw_t + \Delta w + |u+v+w|^\sigma(u+v+w) - |u|^\sigma u - |v|^\sigma v = 0, w(0) = w_0 \in H^1.$$

Define

$$M(t) = \|u\|_{L^\gamma((0,t),W^{1,\rho})} + \|v\|_{L^\gamma((0,t),W^{1,\rho})}.$$

We recall (cf. [12, Corollary 2.3.9]) that

$$\|S(t)w_0\|_{S^s(0,t)} \leqslant \|w_0\|_{H^s}, \quad t > 0.$$

Thus, applying Strichartz estimates on the Duhamel formula for w on a time interval $0 < t < T$,

$$\begin{aligned}\|w\|_{S^s(0,t)} \lesssim \|w_0\|_{H^s} &+ \left(\|u\|^\sigma_{L^\gamma((0,t),B^s_{\rho,2})} + \|v\|^\sigma_{L^\gamma((0,t),B^s_{\rho,2})} + \|w\|^\sigma_{S^s(0,t)}\right)\|w\|_{S^s(0,t)}\\ &+ \|N(u,v)\|_{L^{\gamma'}((0,t),B^s_{\rho',2})}.\end{aligned}$$

where $N(u,v)$ is defined as in (9.1.6). It follows from Lemmata 9.1.3 and 9.1.4 that

$$\|w\|_{S^s(0,t)} \lesssim \|w_0\|_{H^s} + \delta(D) + M(t)^\sigma\|w\|_{S^s(0,t)} + \|w\|^{\sigma+1}_{S^s(0,t)}.$$

Choose T_0 such that

$$M^\sigma(T_0) \lesssim \frac{1}{2}.$$

Then

$$\|w\|_{S^s(0,t)} \lesssim \|w_0\|_{H^s} + \delta(D) + \|w\|^{\sigma+1}_{S^s(0,t)}.$$

An obstruction argument now implies that, if

$$\|w_0\|_{H^s} + \delta(D) < \eta', \quad \eta' < \eta,$$

then w exists (as an H^s solution) up to time T_0 and $\|w(T_0)\|_{H^s} < 2\eta'$. For small enough $\|w_0\|_{H^s}$ and D large, the process can be iterated so that w is defined on $[0,T]$ and $\|w\|_{S^s(0,T)} < \epsilon$. The proof of the global existence is completely analogous to the proof for the critical case. □

To finish this section, consider the weakly coupled nonlinear Schrödinger system

$$\begin{cases} iu_t + \Delta u + k_{11}|u|^{2p}u + k_{12}|v|^{p+1}|u|^{p-1}u = 0 \\ iv_t + \Delta v + k_{22}|v|^{2p}v + k_{12}|u|^{p+1}|v|^{p-1}v = 0 \end{cases}, \quad u, v \in C([0,T), H^1(\mathbb{R}^d)), \qquad \text{(2-NLS)}$$

where $k_{ij} \in \mathbb{R}$ and $1 \leqslant p < 2/(d-2)^+$ (see Part I). Using the standard techniques available for the (NLS), one may show that the initial value problem is locally well-posed for $u_0, v_0 \in H^s(\mathbb{R}^d)$ if $p < 2/(d-2s)^+$ and is conditionally locally well-posed if $p = 2/(d-2s)^+$. We set $T(u_0, v_0)$ as the maximal time of existence of the solution with initial conditions u_0, v_0 and write $(u,v) = (2\,\mathrm{NLS})(u_0, v_0)$. Finally, write

$$\mathcal{GD}_2 = \left\{ (u_0, v_0) \in (H^1(\mathbb{R}^d))^2 : T((2\,\mathrm{NLS})(u_0,v_0)) = \infty, \|(2\,\mathrm{NLS})(u_0,v_0)\|_{(S^1(0,\infty))^2} < \infty \right\}$$

It is easy to check that (2-NLS) has finite speed of disturbance for each component: if u_0 has compact support and B is a set such that $\mathrm{dist}(\mathrm{supp}\, u_0, B) > 0$, then

$$\|u(t)\|_{L^2(B)} \leqslant \frac{2\sup_{s\in[0,t]}\|\nabla u(s)\|_2}{\mathrm{dist}(\mathrm{supp}\, u_0, B)} t.$$

The same is valid for v. Consequently, one may prove the analogous concatenation result:

Proposition 9.1.5. *Set $p = 2/d$ or $p \geqslant 2$. Given two compactly supported initial data $\mathbf{u}_0, \mathbf{v}_0 \in (H^1(\mathbb{R}^d))^2$, a fixed time $T < T(\mathbf{u}_0), T(\mathbf{v}_0)$ and $\epsilon > 0$, there exists $D_T > 0$ such that*

$$T(\mathbf{u}_0 + \mathbf{v}_0(\cdot - y)) > T, \quad |y| > D_T$$

and, taking s such that $p \geqslant 2/(d-2s)$,

$$\|(2\,\mathrm{NLS})(\mathbf{u}_0 + \mathbf{v}_0(\cdot - y)) - (2\,\mathrm{NLS})(\mathbf{u}_0) - (2\,\mathrm{NLS})(\mathbf{v}_0(\cdot - y))\|_{(S^s(0,T))^2} < \epsilon.$$

Moreover, if $\mathbf{u}_0, \mathbf{v}_0 \in \mathcal{GD}$, there exists $D_\infty > 0$ such that $\mathbf{u}_0 + \mathbf{v}_0(\cdot - y) \in \mathcal{GD}$, $|y| > D_\infty$, and

$$\|(2\,\mathrm{NLS})(\mathbf{u}_0 + \mathbf{v}_0(\cdot - y)) - (2\,\mathrm{NLS})(\mathbf{u}_0) - (2\,\mathrm{NLS})(\mathbf{v}_0(\cdot - y))\|_{(S^s(0,\infty))^2} < \epsilon.$$

When $k_{11}, k_{22} < 0$, any compactly supported initial data of the form $\mathbf{u}_0 = (u_0, 0)$ or $\mathbf{v}_0 = (0, v_0)$ is in $\mathcal{GD}_2$: the system (2-NLS) is reduced to a defocusing (NLS). As a consequence, we have

Corollary 9.1.6. *Set $p = 2/d$ or $p \geqslant 2$. Moreover, suppose that $k_{11}, k_{22} < 0$. Given $u_0, v_0 \in H^1(\mathbb{R}^d)$ compactly supported, there exists $D_\infty > 0$ such that $(u_0, v_0(\cdot - y)) \in \mathcal{GD}_2$, for any $|y| > D_\infty$.*

Thus blow-up behaviour can only appear if the initial supports of the two components are sufficiently close to each other. We recall that, if $k_{12} > 0$ is large, blow-up behavior is possible, by the usual Virial argument.

9.2 Unbounded solutions with infinite variance

Consider the initial value problem for the focusing L^2-(super)critical (NLS) over $\mathbb{R}^2$:

$$iu_t + \Delta u + |u|^\sigma u = 0, \quad u(0) = u_0 \in H^1(\mathbb{R}^2), \quad \sigma \geqslant 2.$$

Theorem 9.2.1. *There exists a universal constant $C > 0$ such that the following is true: given $u_0 \in H^1(\mathbb{R}^2)$ real with $E(u_0) < 0$, suppose that there exists a sequence of real functions $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^2)$ such that $E(u_0 + v_n) < 0$ and*

$$\|v_n\|_{H^1} < n \exp\left(-Cn^\theta \left|\frac{V(u_0+v_n)}{E(u_0+v_n)}\right|^{1/2}\right), \quad \theta = \frac{2(\sigma+2)\sigma}{3}. \tag{9.2.1}$$

Then, if the solution u of (NLS) with initial data u_0 is globally defined,

$$\|u\|_{L^\infty((0,\infty);H^1)} = +\infty.$$

Proof. By contradiction, suppose that u is globally defined and that, for some large n,

$$\|u\|_{L^\infty((0,\infty);H^1)} < n.$$

Set $v_0 := v_n$ given by (9.2.1). As a consequence, $\|v_0\|_{H^1} < n$, $V(u_0 + v_0) < \infty$ and $E(u_0 + v_0) < 0$. It follows from the Virial argument that the solution w of (NLS) with initial data $w_0 = u_0 + v_0$ blows-up before

$$T^* := \left|\frac{V(u_0+v_n)}{E(u_0+v_n)}\right|^{1/2}.$$

On the other hand, one may write the equation for the remainder $v = w - u$:

$$iv_t + \Delta v + |u+v|^\sigma(u+v) - |u|^\sigma u = 0, \quad v(0) = v_0.$$

Using Kato's method, we claim that the solution of this Cauchy problem exists up to time

$$T_* := \frac{1}{Cn^\theta} \ln\left(\frac{n}{\|v_0\|_{H^1}}\right).$$

We briefly sketch the required estimates: set

$$\Phi(v)(t) := S(t)u_0 + i\int_0^t S(t-s)\left(|u+v|^\sigma(u+v) - |u|^\sigma u\right)(s)ds,$$

Define $\rho = \sigma + 2$ and γ so that (γ, ρ) is an admissible pair. Then, for any admissible pair (q, r),

$$\begin{aligned}\|\Phi(v)\|_{L^q((0,T);W^{1,r})} &\lesssim \|v_0\|_{H^1} + \||u+v|^\sigma(u+v) - |u|^\sigma u\|_{L^{\gamma'}((0,T);W^{1,\rho'})} \\ &\lesssim \|v_0\|_{H^1} + T^{\frac{1}{\gamma'}-\frac{1}{\gamma}}\|v\|^{\sigma+1}_{L^\gamma((0,T),W^{1,\rho})}\end{aligned} \tag{9.2.2}$$

$$+ \||u|^\sigma(|v| + |\nabla v|)\|_{L^{\gamma'}((0,T),L^{\rho'})} + \||\nabla u||u|^{\sigma-1}|v|\|_{L^{\gamma'}((0,T),L^{\rho'})}.$$

For the last term, Hölder's inequality implies that

$$\||\nabla u||u|^{\sigma-1}|v|\|_{\rho'} \lesssim \|\nabla u\|_2 \|u\|_{2(\sigma+2)}^{\sigma-1} \|v\|_{2(\sigma+2)} \lesssim n^\sigma \|v\|_{2(\sigma+2)}$$

and so, choosing p so that $(p, 2\sigma+4)$ is an admissible pair,

$$\||\nabla u||u|^{\sigma-1}|v|\|_{L^{\gamma'}((0,T),L^{\rho'})} \lesssim n^\sigma T^{\frac{1}{\gamma'}-\frac{1}{p}} \|v\|_{L^p((0,T),L^{2(\sigma+2)})}.$$

The penultimate term in (9.2.2) can be estimated analogously. Notice that $p < \gamma$. With these estimates, one then chooses T so that

$$T^{\frac{1}{\gamma'}-\frac{1}{p}} n^\sigma + T^{\frac{1}{\gamma'}-\frac{1}{\gamma}} \|v_0\|_{H^1}^\sigma < \left(T^{\frac{1}{\gamma'}-\frac{1}{p}} + T^{\frac{1}{\gamma'}-\frac{1}{\gamma}}\right) n^\sigma < T^{\frac{1}{\gamma'}-\frac{1}{p}} n^\sigma = \frac{1}{2C},$$

where C is a universal constant. The fixed point argument then ensures that v exists up to time T and that $\|v(T)\|_{H^1} < 2\|v_0\|_{H^1}$. This process may be iterated for as long as $\|v(kT)\|_{H^1} < 2^k \|v_0\|_{H^1} < n$. Thus v exists up to time

$$T_* = \frac{1}{Cn^\theta} \ln\left(\frac{n}{\|v_0\|_{H^1}}\right)$$

as claimed.

Finally, we remark that the maximal time of existence of v must coincide with that of w. Thus one must have

$$\frac{1}{Cn^\theta} \ln\left(\frac{n}{\|v_n\|_H^1}\right) = T_* < T^* = \left|\frac{V(u_0+v_n)}{E(u_0+v_n)}\right|^{1/2},$$

that is,

$$\|v_n\|_{H^1} > n \exp\left(-Cn^\theta \left|\frac{V(u_0+v_n)}{E(u_0+v_n)}\right|^{1/2}\right)$$

which contradicts the choice of v_n. □

Using Theorem 9.2.1, one may now exhibit several unbounded nonradial solutions of the (NLS).

Example 9.2.1. Suppose that $u_0 \in H^1(\mathbb{R}^2)$ is such that $u_0(x) \sim |x|^{-2}$, $|\nabla u_0(x)| = O(|x|^{-2})$ and $E(u_0) < 0$. The prescribed asymptotic behaviour implies that

$$\int |x|^2 |u_0(x)|^2 dx \sim \int \frac{1}{|x|^2} dx = +\infty.$$

For any given $R > 0$ large, choose a cutoff function $\phi_R \in C^\infty(\mathbb{R}^2)$ so that $\phi_R \equiv 0$ over $B_R(0)$ and $\phi_R \equiv 1$ on $\mathbb{R}^2 \backslash B_{R+1}(0)$. Then, setting $v_R = -\phi_R u_0$,

$$\int |v_R(x)|^2 + |\nabla v_R(x)|^2 dx \lesssim \int_R^\infty \frac{1}{r^3} dr \lesssim \frac{1}{R^2},$$

$$\int |x|^2|u_0+v_R|^2dx \lesssim 1 + \int_1^R \frac{1}{r}dr \lesssim \ln R,$$

and $E(u_0+v_R) \to E(u_0)$. Fixed $n \in \mathbb{N}$, we have, for a sufficiently large $R = R(n)$,

$$\|v_R\|_{H^1} \lesssim \frac{1}{R^2} < n\exp\left(-Cn^\theta \left|\frac{\ln R}{E(u_0+v_R)}\right|^{1/2}\right) \lesssim n\exp\left(-Cn^\theta \left|\frac{V(u_0+v_R)}{E(u_0+v_R)}\right|^{1/2}\right).$$

Thus $v_n = v_{R(n)}$ satisfies the conditions of Theorem 9.2.1 and so the solution u of (NLS) with initial data u_0 is unbounded in H^1.

Example 9.2.2. Take any compactly supported function $f \in H^1(\mathbb{R}^2)$ with $E(f) < 0$. Choose another compactly supported function $g \in H^1(\mathbb{R}^2)$ (for simplicity, take $\operatorname{supp} f, g \subset B_1(0)$). Given any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ with $|x_n| > 2|x_{n-1}|$ and $|x_1| > 3$, define

$$u_0(x) = f(x) + \sum_{i\geqslant 1} \lambda_i g(x-x_i), \quad \lambda_m = \left(V(g(\cdot - x_i))\right)^{-1/2}.$$

and

$$v_n(x) = -\sum_{i>n} \lambda_i g(x-x_i).$$

In this way, it is clear that $V(u_0+v_n) = V(f)+n$. Moreover, for $|x_1|$ large enough, $\|v_n\|_{H^1}$ is small and so $E(u_0+v_n) < E(u_0)/2$, for any n. It is clear that, when $|x_{n+1}| \to \infty$, one has $\|v_n\|_{H^1} \to 0$, since

$$\lambda_m^{-2} = \int_{B_1(x_m)} |x|^2|g|^2 \geqslant (|x_m|^2-1)\|g\|_2^2 \geqslant (4^m|x_1|^2-1)\|g\|_2^2.$$

Then one chooses $R = R(n)$ large such that

$$n\exp\left(-Cn^\theta\left(2\frac{V(f)+n}{E(u_0)}\right)^{1/2}\right) > \|v_n\|_{H^1}, \quad \text{whenever } |x_{n+1}| > R_n.$$

This placement of $\{x_n\}_{n\in\mathbb{N}}$ provides an initial data u_0 and a sequence $\{v_n\}_{n\in\mathbb{N}}$ satisfying the conditions of Theorem 9.2.1.

Appendix A

Essential estimates and linear theory

A.1 Gagliardo-Nirenberg inequality

Given $k > 0$, $p \geqslant 1$, consider the space $\dot{W}^{k,p}(\mathbb{R}^d)$ endowed with the semi-norm

$$\|u\|_{\dot{W}^{k,p}(\mathbb{R}^d)} = \sum_{|\alpha|=k} \left(\int_{\mathbb{R}^d} |D^\alpha u(x)|^p dx \right)^{\frac{1}{p}}.$$

An important feature of these spaces is that they behave well under scalings: if $u_\lambda(x) = u(\lambda x)$, then

$$\|u_\lambda\|_{\dot{W}^{k,p}(\mathbb{R}^d)} = \sum_{|\alpha|=k} \left(\int_{\mathbb{R}^d} \lambda^{kp} |(D^\alpha u)(\lambda x)|^p dx \right)^{\frac{1}{p}} = \lambda^{k-\frac{d}{p}} \|u\|_{\dot{W}^{k,p}(\mathbb{R}^d)}.$$

We define the *scaling index* as the number $l(k,p,d) = l(\dot{W}^{k,p}(\mathbb{R}^d)) := k - d/p$. This notion may also be extended to the case $p = \infty$: we set $l(k,\infty,d) = k$.

Remark A.1. The scaling index may be defined for other function spaces: if $B \subset \mathcal{D}'(\mathbb{R}^d)$ is a vector space, equipped with a semi-norm $\|\cdot\|_B$ such that

$$\|u_\lambda\|_B \leqslant \lambda^b \|u\|_B, \quad u \in B,$$

then one defines $l(B) = b$. For example, $l(\dot{H}^{s,p}(\mathbb{R}^d)) = s - d/p$, for $s \in \mathbb{R}$.

Proposition A.1.1 (Gagliardo-Nirenberg's inequality)**.** *Consider three different pairs (k,p), (m,r), $(0,q) \in \mathbb{N} \times [1,\infty]$ such that*

$$l(k,p,d) = \theta l(m,r,d) + (1-\theta) l(0,q,d), \text{ for some } \theta \in \left[\frac{k}{m}, 1\right].$$

We exclude the case $p = \infty$, $r > 1$ and $\theta = 1$. Then

$$\|u\|_{\dot{W}^{k,p}(\mathbb{R}^d)} \lesssim \|u\|^\theta_{\dot{W}^{m,r}(\mathbb{R}^d)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^d)}, \quad u \in C_c^\infty(\mathbb{R}^d).$$

The relation between the scaling indices can be derived by noticing that the inequality must be preserved under scalings. The restriction $\theta m \geqslant k$ reflects the need of sufficent m derivatives to be able to control k derivatives.

As a consequence of the Gagliardo-Nirenberg inequality,

Theorem A.1.2 (Sobolev injection). *Given $\Omega \subset \mathbb{R}^d$ and $1 \leqslant r \leqslant \infty$,*

1. *if $r < d$, then $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$, for $p \in \left[r, \frac{dr}{d-r}\right]$;*
2. *if $r = d > 1$, then $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$, for $r \leqslant p < \infty$;*
3. *if $r = d = 1$, then $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$, for $r \leqslant p \leqslant \infty$;*
4. *if $r > d$, then $W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^{0,1-d/p}(\bar{\Omega})$, the space of bounded continuous functions which are Hölder continuous of exponent $1 - d/p$.*

If Ω is bounded and has a uniformly Lipschitz continuous boundary, then the injections 1., 2. and 3. are compact for $p \neq \infty$.

A.2 The Schrödinger unitary group

Consider the initial value problem

$$iu_t + \Delta u = 0,\ u(0,x) = u_0(x),\ u = u(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{LS}$$

Using the Fourier transform, it is easy to conclude that, for any $u_0 \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} u(t,x) &= (S(t)u_0)(x) := \mathcal{F}^{-1}(e^{-4\pi^2 i\xi^2 t}\mathcal{F}u_0)(x) \\ &= \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} u_0(y)dy, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \end{aligned} \tag{A.1}$$

is a solution of (LS) in $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^d))$. Since

$$\|u(t)\|_2 = \|e^{-4\pi^2 i\cdot^2 t}\mathcal{F}u_0\|_2 = \|\mathcal{F}u_0\|_2 = \|u_0\|_2,$$

the family $\{S(t)\}_{t\in\mathbb{R}}$ may be extended to a unitary group defined on $L^2(\mathbb{R}^d)$. It follows from classical semigroup theory that $\{S(t)\}_{t\in\mathbb{R}}$ is generated by $i\Delta$ (as an operator defined on $L^2(\mathbb{R}^d)$). Hence, for any $u_0 \in L^2$, the unique solution of (LS) in $C(\mathbb{R}; L^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}; H^{-2}(\mathbb{R}^d))$ is given by (A.1).

It follows directly from (A.1) that

$$\|u(t)\|_\infty \lesssim \frac{1}{t^{d/2}}\|u_0\|_1, \quad u_0 \in \mathcal{S}(\mathbb{R}^d).$$

Applying the Riesz-Thorin interpolation theorem, we have, for any $p \geqslant 2$,

$$\|u(t)\|_p \lesssim \frac{1}{t^{d\left(\frac{1}{2}-\frac{1}{p}\right)}}\|u_0\|_{p'}, \quad u_0 \in \mathcal{S}(\mathbb{R}^d). \tag{A.2}$$

Remark A.1. All these considerations are still valid for the linear hyperbolic Schrödinger equation

$$iu_t + u_{xx} - \Delta_{\mathbf{y}} u = 0,\ u = u(t, x, \mathbf{y}),\ (t, x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}.$$

As in the general ODE theory, the knowledge of the linear solutions allows to solve the inhomogeneous equation

$$iu_t + \Delta u = f,\ u(0) = u_0. \tag{A.3}$$

Indeed, if $f \in C(\mathbb{R}, L^2(\mathbb{R}^d))$, then $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ is a solution of (A.3) in the sense of distributions if and only if u satisfies the Duhamel formula

$$u(t) = S(t)u_0 - i \int_0^t S(t-s) f(s) ds, \quad t \in \mathbb{R}. \tag{A.4}$$

Other combinations of regularities between u and f give the same result. We highlight the following:

Proposition A.2.1 (Duhamel formula)**.** *Suppose that $u_0 \in H^1(\mathbb{R}^d)$ and that $f \in C([0,T], H^{-1}(\mathbb{R}^d))$. If u given by* (A.4) *is in $C([0,T]; H^1(\mathbb{R}^d))$, then it is the unique solution of* (A.3) *in the class $C([0,T]; H^1(\mathbb{R}^d)) \cap C^1((0,T); H^{-1}(\mathbb{R}^d))$.*

A.3 Strichartz estimates

We begin with some heuristic arguments. Given $p \geqslant 2$, we define $s_2(L^p(\mathbb{R}^d)) = s_2(p,d)$ such that

$$l(s_2(p,d), 2, d) = l(0, p, d),\ \text{i.e.,}\ s_2(p,d) = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

The above relation means that $\dot{H}^{s_2(p,d)}(\mathbb{R}^d)$ has the same scaling index as $L^p(\mathbb{R}^d)$. Our heuristic will consist in saying that these two spaces behave in a similar fashion, because their index is the same.

Let $u \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d))$ be a solution of (LS). Since

$$s_2(L^\infty(\mathbb{R})) = 1/2, \quad s_2(L^2(\mathbb{R}^d)) = 0,$$

we say that u has $1/2$ derivatives in time and 0 in space. Now, since u satisfies $iu_t + \Delta u = 0$, one sees that one may convert a given number of time derivatives into twice that number of spatial derivatives. That is, choosing $q \geqslant 2$, this means that u also has $1/2 - 1/q$ derivatives in time and $2/q$ derivatives in space. Converting back into L^p spaces, since

$$l(\dot{H}^{1/2-1/q}(\mathbb{R})) = l(L^q(\mathbb{R})), \quad l(\dot{H}^{2/q}(\mathbb{R}^d)) = l(L^r(\mathbb{R}^d)),\ \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right),$$

this tells us that (possibly) one has $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^d))$. Another way to see the relationship between q and r is to notice that the scaling $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ maps solutions of the (NLS) into solutions of the (NLS). Then any space where the solution may live should behave, under the scaling invariance, in the same way as the space $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$.

Remark A.1. Saying that two spaces behave in a similar fashion based solely on their scaling index is obviously incorrect. However, one should regard these arguments in the opposite direction: if two spaces do *not* have the same scaling index, then they surely behave in different ways.

Definition A.3.1 (Admissible pair)**.** *We say that a pair* (q, r) *is admissible if*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \ 2 \leqslant q, r \leqslant \infty, \ (q, r) \neq (2, \infty).$$

The hidden regularity (derived heuristically) can be made concrete using the linear decay estimate (A.2) and some duality arguments:

Theorem A.3.2 (Strichartz estimates)**.** *Given two admissible pairs* (q, r) *and* (γ, ρ)*, we have*

$$\|S(\cdot)u_0\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|u_0\|_2,$$

$$\left\|\int_{\mathbb{R}} S(-s)f(s)ds\right\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))},$$

and

$$\left\|\int_{s<t} S(t-s)f(s)ds\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \lesssim \|f\|_{L^{\gamma'}(\mathbb{R};L^{\rho'}(\mathbb{R}^d))}.$$

Remark A.2. These inequalities may be seen as a type of Gagliardo-Nirenberg inequality where one does not lose any spatial derivatives through the estimate.

Appendix B

Invariances of the nonlinear Schrödinger equation

The nonlinear Schrödinger equation presents a number of invariances[1] which are decisive for the qualitative study of the equation. In fact, beyond the generic group properties, decay estimates and Strichartz estimates (which do not depend on the nonlinearity), the invariances are the simplest object that distinguishes different nonlinearities and, consequently, different qualitative behaviours.

The nonlinear Schrödinger equation

$$iu_t + \Delta u \pm |u|^\sigma u = 0,\ u = u(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

has the following invariances;

1. Space-time translations: $v(t,x) = u(t+t_0, x+x_0)$;
2. Gauge invariance; $v(t,x) = e^{i\theta} u(x,t)$;
3. Galilean invariance: $v(t,x) = e^{\frac{i}{2}(ax - \frac{i}{4}a^2 t)} u(t, x - at)$;
4. Dilation invariance; $v(t,x) = \lambda^{\frac{2}{\sigma}} u(\lambda^2 t, \lambda x)$;
5. Rotation invariance: $v(t,x) = u(t, Ax)$, A rotation matrix;

In the L^2-critical case, one may find another class of invariances, for which we present the complete formal deduction: given smooth functions $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $g, a, b, f : [0,T) \to \mathbb{R}$, define

$$u(t,x) = v\left(g(t), \frac{x}{b(t)}\right) \exp\left(\frac{ia(t)|x|^2}{4}\right) f(t) \tag{B.1}$$

A tedious but straighfoward calculation shows that

$$iu_t + \Delta u + \lambda |u|^{4/d} u = \Big(i\frac{f}{f'} v - i\frac{b'}{b^2} x \cdot \nabla v + iv_s g' - a'\frac{|x|^2}{4} v + b^{-2}\Delta v + iab^{-1} x \cdot \nabla v$$

[1] Mappings T that act on solutions u of the equation and produce a new solution $v = Tu$.

$$+\frac{iad}{2}v - v\frac{a^2}{4}|x|^2 + \lambda|f|^{4/d}|v|^{4/d}v\Big) \times \exp\left(\frac{ia|x|^2}{4}\right) f$$

If we consider the restrictions

$$(i)\ \frac{f'}{f} = -a\frac{d}{2}, \quad (ii)\ b' = ab, \quad (iii)\ f(0) = b(0) = 1$$

(which imply that $f = b^{-d/2}$), we obtain

$$iu_t + \Delta u + \lambda|u|^{4/d}u = \Big(iv_s g' - \frac{a' + a^2}{4}|x|^2 v + b^{-2}\Delta v + \lambda b^{-2}|v|^{4/d}v\Big) \times \exp\left(\frac{ia|x|^2}{4}\right) f$$

Finally, if we demand that, for some $k \in \mathbb{R}$,

$$(iv)\ g' = b^{-2}, \quad (v)\ a' + a^2 = 4kb^{-4},$$

one arrives at

$$iu_t + \Delta u + \lambda|u|^{4/d}u = \Big(iv_s + \Delta v + \lambda|v|^{4/d}v - k|x|^2 v\Big) \times b^{-2}\exp\left(\frac{ia|x|^2}{4}\right) f.$$

Therefore u is a solution of (NLS) if and only if v is a solution of the (NLS) with a harmonic potential

$$iv_s + \Delta v + \lambda|v|^{4/d}v - k|x|^2 v = 0.$$

Notice that, from (ii) and (v),

$$a'' + 2a'a = -4kb^{-4}\left(4\frac{b'}{b}\right) = -(a' + a^2)4a.$$

Hence

$$a'' + 6aa' + 4a^3 = 0 \tag{B.2}$$

with an initial condition $a(0) = a_0$ and, since $b(0) = 1$, $a'(0) = 4k - a_0^2$. It follows from $(i), (ii), (iii)$ and (iv) that

$$b(t) = \exp\left(\int_0^t a(\tau)d\tau\right),\ f(t) = \exp\left(-\frac{d}{2}\int_0^t a(\tau)d\tau\right),$$

$$g(t) = g(0) + \int_0^t \exp\left(-2\int_0^\rho a(\tau)d\tau\right)d\rho. \tag{B.3}$$

Notice that the presence of $g(0)$ induces only a translation in time, which is a well known-invariance, and so we shall consider

$$(vi)\ g(0) = 0.$$

In conclusion, the transform defined by (B.1) with hypothesis $(i)-(vi)$ is a 2-parameter transform, the *generalized pseudo-conformal transform*, which we denote by $u = \mathcal{T}_{a_0,k}v$. For example, choosing $a_0 = 0$ and $k \in \mathbb{R}^+$, we obtain from (B.2)

$$a(t) = \frac{4kt}{4kt^2+1},\ b(t) = (4kt^2+1)^{1/2},\ g(t) = \frac{1}{\sqrt{4k}}\tan^{-1}(\sqrt{4k}t),\ f(t) = \frac{1}{(4kt^2+1)^{d/4}}.$$

In the special case $k = 1/4$, the transformation (B.1) reads

$$u(t,x) = \left(\mathcal{T}_{0,1/4}v\right)(t,x) = \frac{1}{(1+t^2)^{d/4}}v\left(\tan^{-1}(t), \frac{x}{\sqrt{1+t^2}},\right)e^{i\frac{|x|^2t}{4(1+t^2)}}$$

which is the inverse of the lens transform (see [11], [71]).

On the other hand, if $k = 0$ and $a_0 \in \mathbb{R}$, we get $a' + a^2 = 0$ and so

$$a(t) = \frac{a_0}{1+a_0t},\ b(t) = 1 + a_0t,\ g(t) = \frac{t}{1+a_0t},\ f(t) = (1+a_0t)^{-d/2}.$$

This means that the transformation $v = \mathcal{T}_{a_0,0}u$ is precisely the pseudo-conformal transform and so the generalized pseudo-conformal invariance is the composition of the lens transform with the usual pseudo-conformal invariance.

Remark B.1. For the hyperbolic nonlinear Schrödinger equation

$$iu_t + u_{xx} - \Delta_{\mathbf{y}}u + \lambda|u|^{4/d}u = 0,\ u = u(t,x),\ (t,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d-1}.$$

analogous computations may be carried out: if one defines

$$u(t,x,\mathbf{y}) = v\left(g(t), \frac{x}{b(t)}, \frac{\mathbf{y}}{b(t)}\right)\exp\left(\frac{ia(t)(x^2-|\mathbf{y}|^2)}{4}\right)f(t),$$

with a, b, f, g as before, then u is a solution of (HNLS) if and only if v is a solution of the (HNLS) with a "harmonic" potential

$$iv_s + v_{xx} - \Delta_{\mathbf{y}}v + \lambda|v|^{4/d}v - k(x^2-|\mathbf{y}|^2)v = 0.$$

Finally, given a function space B for which one may define the scaling index $l(B)$ (see Appendix A), we define the notion of B-criticallity for the (NLS): recall that the scaling

$$u \mapsto u_\lambda(t,x) = \lambda^{\frac{2}{\sigma}}u(\lambda^2t, \lambda x), \quad \lambda > 0$$

is an invariance of the (NLS). Then

$$\|u_\lambda(t)\|_B = \lambda^{\frac{2}{\sigma}+l(B)}\|u(\lambda^2t)\|_B.$$

We then say that the (NLS) (or σ) is B-critical if the scaling does not change the norm in B, *i.e.*, $l(B) = -2/\sigma$. If $l(B) < -2/\sigma$ (resp. $l(B) > -2/\sigma$), we say that the equation is B-supercritical (resp. B-subcritical). For example, if $B = L^2(\mathbb{R}^d)$, then $l(B) = -d/2$ and so σ is L^2-critical if $\sigma = 4/d$.

Remark B.2. In a subcritical case, if one chooses $\lambda < 1$, then the time of existence of u_λ increases and its B-norm decreases. This type of dependence is a natural one: smaller solutions should exist for larger times. Indeed, the standard applications of the Banach fixed point theorem to prove local well-posedness imply this dependence. On the other hand, in the supercritical case, one observes an increase in both time of existence and B-norm (which means that larger solutions exist for larger times). This unnatural dependence indicates that the Cauchy problem may be ill-posed for supercritical cases.

Appendix C

Dual dynamical systems

This appendix is dedicated to some observations concerning the theory of invariances and evolution laws for evolution equations. The goal is to present in a clear and practical way these concepts. The main consequence is that one shall be able to obtain evolution laws with trivial computations.

Let $\mathcal{H}$ be a complex Hilbert space, with inner product $\langle\cdot,\cdot\rangle$. Taking $(\cdot,\cdot) = \mathrm{Re}\langle\cdot,\cdot\rangle$, $\mathcal{H}$ becomes a real Hilbert space. Let H, G real functionals defined on a dense subspace $\mathcal{V}$, with continuous derivatives in $\mathcal{H}$. Consider the hamiltonian dynamical system over $\mathcal{V}$ given by

$$i\frac{du}{dt} = H'(u), \tag{SD_H}$$

which we assume to be locally well-posed. The goal is to study the time evolution of G throughout the trajectories of this system. One has

$$\frac{d}{dt}G(u(t)) = \left(G'(u(t)), \frac{du}{dt}(t)\right) = (G'(u(t)), -iH'(u(t))) =: P(u(t)). \tag{C.1}$$

Given $w \in \mathcal{V}$, consider the initial value problem

$$i\frac{dv}{dt} = G'(v), v(0) = w, \tag{SD_G}$$

which we suppose that is locally well-posed, Then

$$\begin{aligned}\frac{d}{dt}H(v(t)) &= (H'(v(t)), \frac{dv}{dt}(t)) = (H'(v(t)), -iG'(v(t)))\\ &= -(G'(v(t)), -iH'(v(t))) = -P(v(t)).\end{aligned}$$

Hence

$$P(w) = -\frac{d}{dt}H(v(t))\Big|_{t=0}, \tag{C.2}$$

and so the evolution of G through the trajectories of H can be determined by computing the evolution of H through the trajectories of G. In this sense, the dynamics are dual to each other.

In most applications, the dynamical system (SD_G) is easily solvable, while (SD_H) is a complex system. Moreover, since (SD_G) is independent of H, one only needs to solve it once to study the evolution of G through *any* system (SD_H),

We now give some ilustrative examples. We consider $\mathcal{H} = L^2(\mathbb{R}^d)$, endowed with the usual inner product. We do not care about the subspace $\mathcal{V}$: it should be clear in each case that it exists.

- Evolution of the L^2 norm: in this case, $G(u) = \frac{1}{2}\|u\|_2^2$. Then system (SD_G) is
$$i\frac{dv}{dt} = v,\ v(0) = w$$
which has the explicit solution $v(t) = e^{-it}w$. Hence, to determine the evolution of the L^2 norm, it suffices to compute $(H(e^{-it}w))'(0)$. If H is gauge invariant, this derivative is null and so the L^2 norm is conserved.

- As it is well known, in a Hamiltonian system, the Hamiltonian is a conserved quantity. This is trivial: taking $G = H$,
$$\frac{d}{dt}H(u(t)) = P(u(t)) = -\frac{d}{dt}H(u(t))$$
and so $\frac{d}{dt}H(u(t)) = 0$.

- The (NLS) as an Hamiltonian system: one of the most essential properties of the (NLS) is that it can be written as
$$i\frac{du}{dt} = E'(u), \quad E(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{\sigma+2}\|u\|_{\sigma+2}^{\sigma+2}.$$
Considering the previous examples, we obtain directly the conservation of mass $\|u\|_2^2$ and the conservation of energy E.

- Linear momentum: setting $G(u) = \frac{1}{2}\operatorname{Im}\int u\bar{u}_x$, system (SD_G) is
$$i\frac{dv}{dt} = iv_x,\ v(0) = w$$
whose solution is $v(x,t) = w(x+t)$. If H is translation invariant, $(H(w(x+t))'(0) = 0,\ \forall w \in \mathcal{V}$, and momentum is conserved. Otherwise, take, for example,
$$H(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|xu\|_2^2 - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2},$$
(which corresponds to the nonlinear Schrödinger equation with an harmonic potential),
$$\frac{d}{dt}H(w(x+t))\Big|_{t=0} = \frac{d}{dt}\frac{1}{2}\|(x-t)w\|_2^2\Big|_{t=0} = -\int x|w|^2$$

and so, if u is a solution of (SD_H),

$$\frac{1}{2}\frac{d}{dt}\operatorname{Im}\int u(t)\bar{u}_x(t) = \int x|u(t)|^2.$$

Notice that, by the classical method, one would multiply the equation by $i\bar{u}_x$ and integrate by parts twice.

- Virial's identity for the (NLS): considering $G(u) = \frac{1}{2}\|xu\|_2^2$, the solution of (SD_G) is $v(t) = e^{-it|x|^2}w$. Hence

$$-P(w) = \frac{d}{dt}H(e^{-it|x|^2}w)\Big|_{t=0} = \frac{d}{dt}\int |-2itxw + \nabla w|^2\Big|_{t=0} = 2\operatorname{Im}\int xw\nabla\bar{w}$$

and so, if u is a solution of (NLS),

$$\frac{d}{dt}\|xu\|_2^2 = -4\operatorname{Im}\int xu(t)\nabla\bar{u}(t).$$

Now we would like to compute the second derivative of G. To that end, we shall take P as the new G and proceed as before: if $G = \operatorname{Im}\int xw\nabla\bar{w}$, (SD_G) becomes

$$i\frac{dv}{dt} = i(2x\cdot\nabla v + dv)$$

with the explicit solution $v(t) = e^{dt}w(e^{2t}x)$ (clearly connected to the scaling invariance of (NLS)). Since

$$\begin{aligned}\frac{d}{dt}H(v(t))\Big|_{t=0} &= \frac{d}{dt}\left(\frac{e^{(2d+4-2d)t}}{2}\|\nabla w\|_2^2 - \frac{e^{(d(\alpha+2)-2d)t}}{\alpha+2}\|w\|_{\alpha+2}^{\alpha+2}\right)\Big|_{t=0}\\ &= 2\|\nabla w\|_2^2 - \frac{d\alpha}{\alpha+2}\|w\|_{\alpha+2}^{\alpha+2},\end{aligned}$$

we finally arrive at

$$\frac{d^2}{dt^2}\|xu\|_2^2 = -4\frac{d}{dt}\operatorname{Im}\int xu(t)\nabla\bar{u}(t) = 8\|\nabla u\|_2^2 - \frac{4d\alpha}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}.$$

We point out that the classical method involves many more computations, whose level of complexity is certainly higher than these quite simple computations.

We now make a series of observations:

Remark C.1. In this framework, Noether's theorem becomes trivial: if H is invariant throughout the orbits of (SD_G), then $P\equiv 0$ and so G is conserved by (SD_H). On the other hand, if G is conserved, $P\equiv 0$ and so H must be invariant by the action of (SD_G).

Remark C.2. These considerations are not restricted to Hamiltonian dynamical systems: one may also consider the system

$$\frac{du}{dt} = H'(u). \tag{C.3}$$

In this case, the dual dynamical system is

$$\frac{dv}{dt} = G'(v),\ v(0) = w \tag{C.4}$$

and there is no change in the sign of P (see (C.2)). This allows the computation of evolution laws for more general systems, such as the heat equation, Moreover, suppose that for some flow $v(t;w)$ of type (C.4), the functional H decreases throughout its trajectories. Hence G also decreases by the action of (C.3), which means that G is a Lyapunov functional. Hence, finding Lyapunov functionals is equivalent to finding ways to decrease H. For these systems, a trivial example is $G = -H$.

Remark C.3. The technique is additive: considering a system of the form

$$\frac{du}{dt} = \sum c_j H_j'(u), \quad c_j \in \{1, -i\},$$

we define P as in (C.1). Letting P_j be the functional related to

$$\frac{du}{dt} = c_j H_j'(u),$$

one easily checks that $P = \sum P_j$. With this observation, one may decompose the dynamical system and compute the derivative for each one individually. A system which falls under this scenario is the complex Ginzburg-Landau equation

$$u_t = e^{i\theta}\Delta u + e^{i\gamma}|u|^\alpha u, \quad \theta, \gamma \in \mathbb{R},$$

with

$$H_1(u) = -\frac{\cos\theta}{2}\|\nabla u\|_2^2 + \frac{\cos\gamma}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}, \quad c_1 = 1$$

and

$$H_2(u) = \frac{\sin\theta}{2}\|\nabla u\|_2^2 - \frac{\sin\gamma}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}, \quad c_2 = -i.$$

Remark C.4. Suppose that G has a parameter $\theta \in \mathbb{R}$, $G = G(\theta, u)$. In this case, one may consider the auxilliary system

$$iv_\theta = G'(\theta, u), v(0) = w.$$

where G' is the derivative in u. Setting

$$P(\theta, z) = (H'(z), -iG'(\theta, z)),$$

one has

$$\frac{d}{d\theta}H(v(\theta))\Big|_{\theta=0} = P(0,w), \quad \frac{d}{dt}G(0,u(t)) = -P(0,u(t)).$$

Notice that one could replace $\theta = 0$ with $\theta = \theta_0$ by considering $\tilde{G}(\theta,u) = G(\theta-\theta_0,u)$. As an example, take the (NLS) in dimension two. This system is invariant by rotations on the space variables:

$$u(t,x,y) \mapsto u(t, x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta) =: v(t,x,y;\theta).$$

Observe that

$$iv_\theta = i[v_x \ v_y]\begin{bmatrix} -x\sin\theta & y\cos\theta \\ -x\cos\theta & -y\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Since (NLS) is invariant through the action of this dynamical system, we know that, if there exists $G = G(\theta,v)$ with $iv_\theta = G(\theta,v)$, then $G(\theta,\cdot)$ will be conserved by the (NLS) flow. After some computations, we conclude that, for $\theta = 0$,

$$G(0,v) = \mathrm{Im}\int (yv_x - xv_y)\bar{v}$$

is a conserved quantity.

Remark C.5. The fact that P determines the evolution of the Hamiltonian in the dual dynamical system was used in [37] to conclude the instability of ground-states for the (NLS) with a magnetic potential. It would be interesting to find more situations where this duality could be used to conclude more properties of the system at hand. For example, the blowup dynamics for the (NLS) could be studied by duality, taking either $G_1(u) = \|\nabla u\|_2^2$ or $G_2(u) = \|u\|_{\alpha+2}^{\alpha+2}$, and understanding how the (NLS) energy evolves through the flow of G_1 or G_2.

Appendix D

Injectivity of the plane wave transform

In this appendix, we give another proof of

$$Tf \equiv 0 \Rightarrow f \equiv 0.$$

This proof is based on arguments that do not involve the expression of T in terms of the Fourier transform. For the sake of simplicity, we shall assume that f is good enough so that all integrations are automatically justified.

Proposition D.0.1. *Given $f \in C(\mathbb{R}^2)$ with exponential decay at infinity, if $Tf \equiv 0$, then $f \equiv 0$.*

Proof. Step 1. Reduction to radial case. Since $Tf \equiv 0$,

$$\int f(x - cy, c)dc = 0, \ \ (x, y) \in \mathbb{R}^2.$$

Then, given any $h \in \mathbb{R}$,

$$\int_0^1 \int f(x + hz - cy, c)dcdz = 0,$$

which means that the integration of f over any non-vertical strip is null. Define

$$\tilde{f}(z, c) = \frac{1}{2\pi} \int_0^{2\pi} f(A_\theta(z, c))d\theta, \quad A_\theta(z, c) = (c\cos\theta - z\sin\theta, c\sin\theta + z\cos\theta)$$

Then, for any $\epsilon, \delta > 0$ and any strip $S =]\epsilon - \delta, \epsilon + \delta[\times \mathbb{R}$,

$$\int_S \tilde{f}(z, c)dcdz = \frac{1}{2\pi} \int_0^{2\pi} \int_S f(A_\theta(z, c))dcd\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_{A_\theta^{-1}S} f(z, c)dcdz = 0,$$

since the rotation of a strip is still a strip. In particular, the function

$$\epsilon \mapsto \int \tilde{f}(\epsilon, c)dc$$

is zero. Since $\tilde{f}$ is a radial function, $\tilde{f}(\epsilon, c)$ is an even function, and so

$$\int_0^\infty \tilde{f}(\epsilon, c)dc = 0, \ \epsilon > 0.$$

Hence

$$0 = \int_0^{2\pi} \int_0^\infty \tilde{f}(\epsilon, c)dcd\theta = \int_{\mathbb{R}^2 \setminus B_\epsilon(0)} \frac{\tilde{f}(z, c)}{\sqrt{c^2 + z^2 - \epsilon^2}} dzdc$$

which implies, in polar coordinates, that

$$\int_\epsilon^\infty \frac{\tilde{f}(r)r}{\sqrt{r^2 - \epsilon^2}} dr = 0, \ \epsilon > 0. \tag{D.1}$$

Step 2. $\tilde{f}$ must be zero. We write $\tilde{f} = f^+ - f^-$, with $f^+, f^- \geqslant 0$ and $f^+ f^- \equiv 0$. Fix $a \in \mathbb{R}^+$. W.l.o.g., let $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ be a strictly increasing sequence with $a_0 = a$ and such that

$$\tilde{f}\big|_{[a_{2k}, a_{2k+1}]} \geqslant 0, \quad \tilde{f}\big|_{[a_{2k+1}, a_{2k+2}]} \leqslant 0, k \in \mathbb{N}. \tag{D.2}$$

There are two possibilities: either a_n is bounded or not. We consider the case where a_n is not bounded, the other being quite similar. Define

$$h_n(r) = \frac{r}{\sqrt{r^2 - a_n^2}} \mathbb{1}_{(a_n, +\infty)}$$

and

$$S_N(r) = \sum_{n=0}^N (-1)^n h_n(r), N \in \mathbb{N}.$$

Since $h_n(r) \leqslant h_{n+1}(r)$ for $r > a_{n+1}$, it is easy to check that, for N even,

$$S_N\big|_{(a_{2k}, a_{2k+1})} > 0, \quad S_N\big|_{(a_{2k+1}, a_{2k+2})} < 0, \quad S_N\big|_{(a_N, \infty)} > 0, \quad k \in \mathbb{N}, 2k + 2 \leqslant N \tag{D.3}$$

This implies that, for all N even, $f^+ S_N \geqslant 0$. On the other hand, the same reasoning shows that, for all N odd, $f^- S_N \leqslant 0$. Given $k \in \mathbb{N}$, it follows from (D.1) and from the decay of f that

$$\begin{aligned} \left| \int_0^\infty f^+ S_{2k} - \int_0^\infty f^- S_{2k+1} \right| &= \left| \int_0^\infty \sum_{n=1}^{2k} (-1)^n (f^+ - f^-) h_n + \int_0^\infty f^- h_{2k+1} \right| \\ &= \left| \sum_{n=1}^{2k} (-1)^n \int_0^\infty f h_n + \int_0^\infty f^- h_{2k+1} \right| \\ &= \left| \int_0^\infty f^- h_{2k+1} \right| \to 0, \ k \to \infty \end{aligned}$$

Since, for any k,

$$\int_0^\infty f^+ S_{2k} \geqslant 0, \int_0^\infty f^- S_{2k+1} \leqslant 0,$$

it follows that

$$\int_0^\infty f^+ S_{2k},\ \int_0^\infty f^- S_{2k+1} \to 0,\ \ k \to \infty.$$

Set $S(r) = \lim S_N(r)$. By Monotone Convergence Theorem, the functions f^+S and $-f^-S$ are integrable, positive and have integral over $[0, \infty)$ equal to 0. Hence $f^+S, f^-S \equiv 0$. It follows from (D.2) and (D.3) that $f^+(r), f^-(r) = 0$ for $r > a$. Since $a > 0$ is arbitrary, we conclude that $\tilde{f} \equiv 0$.

Step 3. Conclusion. Since $\tilde{f}$ is the average of f over any circle centered at the origin and $\tilde{f} \equiv 0$, it follows that $f(0, 0) = 0$. To prove that $f(z_0, c_0) = 0$ for any $(z_0, c_0) \in \mathbb{R}^2$, consider

$$f_{(z_0,c_0)}(z, c) = f(z + z_0, c + c_0).$$

Then $Tf_{(z_0,c_0)} \equiv 0$ and therefore, by the previous steps, $f(z_0, c_0) = f_{(z_0,c_0)}(0, 0) = 0$. $\square$

REMARK D.1. It follows directly from this proof that the integral of f over the collection of all strips in $\mathbb{R}^2$ determines uniquely the values of f. This implies that the integral of f over all strips determines the integral of f over all balls. This is not trivial at all, since the value of the latter cannot be obtained from the first ones using algebraic set relations. Furthermore, it is easy to check that, if one starts with a collection of strips with a finite number of possible directions, such a result is no longer valid.

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